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A SIMPLE PROOF OF THE JORDAN-CURVE THEOREM

By K. VENKATACHALIENGAR (*Bangalore, India*)

[Received 8 March 1937; in revised form 29 June 1937]

THE following simple proof of the theorem is founded on the idea that, if a closed Jordan-curve j contains a line* as part of it, then it divides the plane into at least two domains. Repeatedly using this the proof of the theorem is obtained.

LEMMA 1. *If a line $\overrightarrow{AB}$ is part of j, it divides the plane into at least two domains.*

Proof. Let C and D be two points such that CD cuts j at a single interior point of AB; then C and D belong to two different domains determined by j. For suppose on the contrary that C and D can be joined by a polygon-chain π not meeting j. We can assume that this does not meet CD in any other points. [This can be achieved, if necessary, by taking two other points, instead of C and D, on CD on opposite sides of AB.] Then π and CD form a polygon. A and B must belong to two different domains determined by this polygon since AB meets its boundary only once; but A and B are connected by means of the continuous curve $\overrightarrow{BA} = j - \overrightarrow{AB}$, without meeting the boundary of the polygon. This is absurd and hence j does determine at least two distinct domains in the plane.

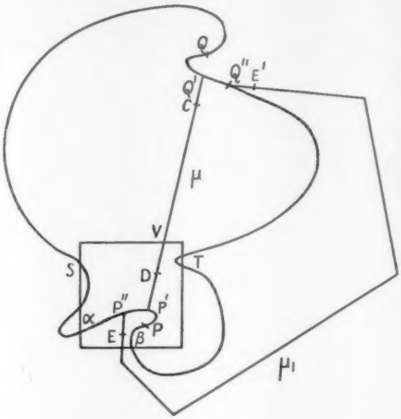
LEMMA 2. *The boundary of any domain determined by j is the complete curve j itself.*

Proof. Suppose there is a domain R whose boundary is part of j. As the boundary of any domain is closed, there is an arc $c_1 = \overrightarrow{PQ}$ of j such that P and Q are boundary points of R and the open arc $\overrightarrow{PQ}$ is free of the boundary points of R. Let $c_2 = j - c_1 =$ arc $\overrightarrow{QP}$. Describe round P a square K (of boundary k), which excludes Q (see figure). Let α and β be the first points of k on j which are obtained as we proceed along j on both sides of P. Let D be a point of R inside K which is also inside a square round P which excludes $j - \overrightarrow{\alpha P \beta}$.† Let

* By 'a line' I shall always mean a straight line.

† I have to thank Mr. J. H. C. Whitehead for this suggestion.

C be a point near Q which is chosen in a similar manner. (The first square round Q must not meet k.) Let Q' and P' be the first points of j on CQ and DP respectively. Then P' and Q' are points of c_2 since the open arc c_1 does not contain any boundary-point of R. Let μ be a polygon-chain in R which connects C and D; μ will meet k



since D is inside K and C outside. Let V be the first point of intersection when we proceed along $\overrightarrow{CD} = \mu$. Let $\overrightarrow{SVT}$ be that part of k which is such that S and T are the only points of j on it. (C, S and T may coincide, but the following argument is applicable in that case also.) Now μ meets k in a finite number of points. We can assume, by altering μ if necessary, that the number of points of intersection of μ and $\overrightarrow{STV}$ is either zero or one. If it is zero, then μ will meet k at one other point at least; and the construction is repeated. Finally, we can assume that $\overrightarrow{SVT}$ meets μ only at V. S and T are points on the boundary of R and hence are points on the open arc c_2. Now on account of the choice of C and D, S and T do not separate P' and Q' on j. Now $\overrightarrow{SVT}$, together with that arc $\overrightarrow{ST}$ of j, which does not contain P, P', Q, Q', forms a Jordan-curve with a line as part of it; hence it determines at least two domains lying on opposite sides of ST. P' and Q' will belong to different domains since μ meets $\overrightarrow{SVT}$ only once; but P' and Q' are connected by means of the

Jordan-arc $j - \overrightarrow{Q'P'}$ (where $\overrightarrow{Q'P'}$ is to be understood as part of c_2). As this contradicts the preceding, we obtain the proof of the lemma.

If R is any region bounded by j, and if S and T are defined as above, P and Q now being any points on j, it follows that S and T separate P and Q on j, and therefore separate the arc $\alpha P \beta$ from the corresponding arc containing Q. For there would otherwise be an arc $P'Q'$ of j which did not contain S or T, and we should have the same contradiction as that which led to the proof of Lemma 2.

From this we immediately deduce that, if a Jordan-curve has a line as part of it, then it divides the plane into exactly two domains, each lying on opposite sides of the line part of the boundary; for every domain has a boundary-point on the line and there can at most be two domains meeting along the line part of j.

LEMMA 3. *j divides the plane into exactly two domains.*

Proof. First, we prove that it divides the plane into at least two domains. Let R_1 and R_2 be the two domains that are formed by $\overrightarrow{SVT}$ and part of j as in Lemma 2. Let D belong to R_1 (see figure), then C belongs to R_2. Take a point E of R_2 inside the smaller square† round P; join EP, and let P'' be the first point of j on $\overrightarrow{EP}$. Now we prove that E and D belong to different domains determined by j. Suppose on the contrary that μ' is a polygon-chain connecting D and E which does not meet j. We can assume, altering E and D if necessary, that μ' does not meet EP'' or DP' again. Now μ', EP'', the arc $\overrightarrow{P''P'}$ of j which is inside K, and $P'D$ form a closed Jordan-curve j' with a line as part of it. As E belongs to R_2, and D to R_1, μ' should meet $\overrightarrow{SVT}$ in an odd number of points, and hence S and T should belong to two different domains determined by j'. But S and T are connected by means of that part of j which does not contain $\overrightarrow{P''P'}$, i.e. by a continuous curve which does not meet j'. This is a contradiction and hence E and D belong to different domains determined by j.

Now, if there is another domain determined by j apart from the two domains which contain D and E respectively, let F be a point of it inside the smaller square round P. Let $\overrightarrow{FP'''P}$ be a line such that P''' is the first point of j on $\overrightarrow{FP}$. Let C, E', F' be similar points of

† See Lemma 2.

the three domains in the smaller square round Q. Let μ_1 and μ_2 (not shown in the figure) be polygon-chains in the respective domains connecting E, E', and F, F' respectively. Let Q'', Q''', be similarly obtained and μ_1 and μ_2 be assumed not to meet EP'', $E'Q''$, etc. Then μ_1, $E'Q''$, the arc $Q''Q'''$ inside the larger square round Q, $Q'''F'$, μ_2, FP''', the arc $P'''P''$ inside K, and $P''E$ form a closed Jordan-curve j'' with a line as part of it. Hence it divides the plane into two domains. Neither μ_1 nor μ_2 can meet $\overrightarrow{SVT}$, since that would contradict the fact that D belongs to a domain which is different from the domains to which E and F belong. For this reason, and since S and T separate P and Q on j, μ (of Lemma 2) and the open arcs of the Jordan-curves that are obtained from j by removing the parts $\overrightarrow{P''P'''}$ and $\overrightarrow{Q''Q'''}$ of j'' are in one of the domains determined by j''. Hence we can join μ_1 and μ_2 in the other domain determined by j'' without meeting any point of j. Hence there are exactly two domains determined by j. This completes the proof of the theorem.

RAMANUJAN AND THE THEORY OF FOURIER TRANSFORMS

By G. H. HARDY (*Cambridge*)

[Received 9 August 1937]

1. DURING 1913 Ramanujan held a research studentship in the University of Madras, and submitted to the University three quarterly reports on the progress of his researches.* He was occupied mainly with definite integrals, and in particular with the formula

$$\int_0^\infty x^{s-1}\{\phi(0)-x\phi(1)+x^2\phi(2)-...\}\,dx = \frac{\pi}{\sin s\pi}\phi(-s)$$

and its developments and corollaries.† He had no real proofs either of this or of any other of his formulae. All of them are valid under appropriate conditions; but it is sometimes not at all obvious what these conditions are, and nearly all of the formulae are worth a careful analysis.

The two which I examine here are

$$\int_0^\infty \left\{\phi(0)-\frac{\phi(1)}{1!}t+\frac{\phi(2)}{2!}t^2-...\right\}\cos xt\,dt$$
$$= \phi(-1)-\phi(-3)x^2+\phi(-5)x^4-..., \quad (1.1)$$

$$\int_0^\infty \left\{\phi(0)-\frac{\phi(1)}{1!}t+\frac{\phi(2)}{2!}t^2-...\right\}\sin xt\,dt$$
$$= \phi(-2)x-\phi(-4)x^3+\phi(-6)x^5-.... \quad (1.2)$$

2. Ramanujan had no doubt learnt Fourier's integral theorem from Carr,‡ and he had found many of the formal developments of this and other reciprocities. For example, as I stated in a recent lecture,§ 'he had most of the formal ideas which underlie the recent work of Watson, and of Titchmarsh and myself,|| on "Fourier kernels" and "reciprocal functions"'. Thus he knew that, if

$$G(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(t)\cos xt\,dt$$

* I owe my knowledge of the contents of these reports to Prof. Watson, who included them in his manuscript copy of Ramanujan's note-books.

† See Hardy (**2**) for a proof of this formula under the most obvious set of conditions. ‡ Carr (**1**), vol. 2, 399–403. § Hardy (**3**).

|| Hardy and Titchmarsh (**4, 5**), Watson (**7**). See also Titchmarsh (**6**), Chs. VIII and IX.

is the Fourier cosine transform of $F(x)$, and

$$f(s) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty F(x)x^{s-1}\,dx, \quad g(s) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty G(x)x^{s-1}\,dx$$

are the 'Mellin transforms' of $F(x)$ and $G(x)$, then

$$g(s) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(s)\cos\tfrac{1}{2}s\pi f(1-s)$$

or

$$\frac{g(s)}{2^{\frac{1}{2}s}\Gamma(\frac{1}{2}s)} = \frac{f(1-s)}{2^{\frac{1}{2}(1-s)}\Gamma\{\frac{1}{2}(1-s)\}};$$

and he was very much interested in other 'integral translations' of the Fourier reciprocity.

In particular, Ramanujan used the formulae (1.1) and (1.2) as a basis for a heuristic theory of Fourier transforms (a theory, naturally, valid only under very narrow conditions).

Suppose, for example, that $\phi(u)$ is an integral function, and that

$$\phi(1) = \phi(3) = \phi(5) = \ldots = 0, \tag{2.1}$$

so that

$$\sum_0^\infty \frac{(-1)^n\phi(n)}{n!}x^n = \sum_0^\infty \frac{\phi(2m)}{2m!}x^{2m}$$

is even; and write

$$\phi_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1+u)\cos\tfrac{1}{2}u\pi\phi(-u-1). \tag{2.2}$$

Then $\phi_1(u)$ is also integral.* Also

$$\phi_1(1) = \phi_1(3) = \phi_1(5) = \ldots = 0,$$

$$\phi_1(2n) = (-1)^n\sqrt{\left(\frac{2}{\pi}\right)}2n!\phi(-2n-1),$$

$$\begin{aligned}\phi_1(-2n-1) &= \sqrt{\left(\frac{2}{\pi}\right)}\phi(2n)\lim_{\epsilon\to 0}\{\Gamma(-2n+\epsilon)\cos\tfrac{1}{2}(2n+1-\epsilon)\pi\} \\ &= (-1)^n\sqrt{\left(\frac{\pi}{2}\right)}\frac{\phi(2n)}{2n!}.\end{aligned}$$

Hence, if we write

$$L(x) = \sum_0^\infty \frac{\phi(2n)}{2n!}x^{2n}, \qquad M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_0^\infty (-1)^n\phi(-2n-1)x^{2n}, \tag{2.3}$$

* The poles of $\Gamma(1+u)$ at $u = -1, -3, -5, \ldots$ are cancelled by zeros of $\cos\frac{1}{2}u\pi$, and those at $u = -2, -4, -6, \ldots$ by zeros of $\phi(-u-1)$.

and denote by L_1 and M_1 the functions derived similarly from ϕ_1, then

$$L_1(x) = M(x), \qquad M_1(x) = L(x);$$

and (1.1) and the corresponding formula with ϕ_1 become

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty L(t)\cos xt\,dt = M(x), \qquad \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty M(t)\cos xt\,dt = L(x),$$

in accordance with Fourier theory.

For example, if

$$\phi(u) = 2^{-1-\frac{1}{2}u}\frac{\Gamma(-\frac{1}{2}u)}{\Gamma(-u)} = \frac{2^{\frac{1}{2}u}\sqrt{\pi},}{\Gamma(\frac{1}{2}-\frac{1}{2}u)},$$

all the conditions are satisfied, and

$$L(x) = \sum_0^\infty \frac{(-1)^n}{n!}(\tfrac{1}{2}x^2)^n = e^{-\frac{1}{2}x^2}.$$

Also

$$\phi_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1+u)\cos\tfrac{1}{2}u\pi\,.\,\frac{2^{\frac{1}{2}u-\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{1}{2}u)}{\Gamma(1+u)} = \frac{2^{\frac{1}{2}u}\sqrt{\pi}}{\Gamma(\frac{1}{2}-\frac{1}{2}u)} = \phi(u),$$

and $M(x) = L(x)$; so that (1.1) expresses the 'self-reciprocal' property of $e^{-\frac{1}{2}x^2}$.

On the other hand, there are many familiar formulae which are not to be accounted for as cases of (1.1). Thus

$$\int_0^\infty e^{-t}\cos xt\,dt = \frac{1}{1+x^2}, \qquad \int_0^\infty \frac{\cos xt}{1+t^2}\,dt = \tfrac{1}{2}\pi e^{-|x|}.$$

The first of these is a case of (1.1) when $-1 < x < 1$, with $\phi(u) = 1$. But then

$$\phi_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1+u)\cos\tfrac{1}{2}u\pi$$

is not an integral function, and we cannot account in this way for the second formula (as is obvious because $e^{-|x|}$ is not expansible as a power-series in x).

3. I shall now obtain sufficient conditions for the validity of Ramanujan's formula. The results could be deduced from more general theories (and, in particular, from that of Titchmarsh and myself); but it is interesting to prove them directly by the methods of 'classical' analysis.

THEOREM 1. *Suppose that $\delta > 0$, and that $\phi(u)$ is an analytic function of*

$$u = v+iw$$

which satisfies the following conditions:

(1) *$\phi(u)$ is an integral function*;

(2) $|\phi(u)| < C|\Gamma(1+u)|e^{Pv+A|w|}$,

where $A < \pi$, for $v \geqslant -\delta$;

(3) $|\phi(u)| < De^{-Qv+B|w|}$,

where $B < \frac{1}{2}\pi$, for $v \leqslant -\delta$. Then

$$\sum_0^\infty \frac{(-1)^n\phi(n)}{n!}t^n \tag{3.1}$$

is convergent for $-e^{-P} < t < e^P$, and represents an analytic function $L(t)$ regular for all positive t; and

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty L(t)e^{-ixt}\,dt = M(x), \tag{3.2}$$

where
$$M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_0^\infty \phi(-n-1)(-ix)^n, \tag{3.3}$$

for $-e^{-Q} < x < e^{-Q}$; so that Ramanujan's formulae (1.1) *and* (1.2) *are true for $-e^{-Q} < x < e^{-Q}$.*

If
$$\psi(u) = \frac{\phi(u)}{\Gamma(1+u)}, \tag{3.4}$$

then $\psi(u)$ is integral,

$$\psi(-1) = \psi(-2) = \ldots = 0,$$

and
$$|\psi(u)| < Ce^{Pv+A|w|} \tag{3.5}$$

for $v \geqslant -\delta$.*

Suppose now that

$$-1 < -\delta < \kappa < 0, \qquad 0 < t < e^{-P}. \tag{3.6}$$

Then a simple application of Cauchy's Theorem shows that

$$L(t) = -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\psi(u)t^u\,du, \tag{3.7}$$

* It would be sufficient to suppose that

$$|\psi(u)| < He^{K|u|}$$

for $v \geqslant -\delta$ and some H and K, and that (3.5) is true for $u = v > 0$ and for $v = -\delta+iw$. It then follows, by a simple argument of 'Phragmén-Lindelöf' type, that (3.5) is true as stated. A corresponding change may be made in conditions (2) and (3) of the theorem.

where t^u has its principal value. This is true, first, for $0 < t < e^{-P}$, but the integral is uniformly convergent in any interval

$$0 < \tau \leqslant t \leqslant T < \infty$$

and gives the analytic continuation of $L(t)$ for all positive t.

It follows that, if x is real and not 0,

$$\int_{\tau}^{T} L(t)e^{-ixt}\,dt = -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\psi(u)\,du \int_{\tau}^{T} e^{-ixt}t^u\,dt,$$

for $0 < \tau < T < \infty$. Also

$$\left|\int_{0}^{\tau} e^{-ixt}t^u\,dt\right| \leqslant \int_{0}^{\tau} t^\kappa\,dt = \frac{\tau^{1+\kappa}}{1+\kappa};$$

and
$$\int_{T}^{\infty} e^{-ixt}t^u\,dt = \frac{e^{-ixT}}{ix}T^u + \frac{u}{ix}\int_{T}^{\infty} e^{-ixt}t^{u-1}\,dt,$$

so that

$$\left|\int_{T}^{\infty} e^{-ixt}t^u\,dt\right| \leqslant \frac{T^\kappa}{|x|} + \frac{|u|}{|x|}\int_{T}^{\infty} t^{\kappa-1}\,dt = \left(1+\frac{|u|}{|\kappa|}\right)\frac{T^\kappa}{|x|}.$$

Hence

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\psi(u)\,du \int_{0}^{\tau} e^{-ixt}t^u\,dt, \qquad \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\psi(u)\,du \int_{T}^{\infty} e^{-ixt}t^u\,dt$$

are convergent and tend to 0 when $\tau \to 0$ and $T \to \infty$; and

$$\int_{0}^{\infty} L(t)e^{-ixt}\,dt = -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\psi(u)\,du \int_{0}^{\infty} e^{-ixt}t^u\,dt$$

$$= -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi}\phi(u)(ix)^{-u-1}\,du. \qquad (3.8)$$

Here
$$(ix)^{-u-1} = \exp\{-(u+1)(\log x+\tfrac{1}{2}\pi i)\}$$

if $x > 0$, and
$$(ix)^{-u-1} = \exp\{-(u+1)(\log|x|-\tfrac{1}{2}\pi i)\}$$

if $x < 0$, the logarithms having their real values.

Now
$$|\phi(u)| = O(e^{-Qv+B|w|})$$

and
$$(ix)^{-u-1} = O(e^{-v\log|x|+\frac{1}{2}\pi|w|})$$

for $v \leqslant -\delta$; and $B+\frac{1}{2}\pi < \pi$. Hence, if $Q+\log|x| < 0$, i.e. if

$-e^{-Q} < x < e^{-Q}$, we can evaluate the last integral in (3.8) by another application of Cauchy's Theorem; and

$$\int_0^\infty e^{-ixt}L(t)\,dt = \sum_1^\infty \phi(-n)(-ix)^{n-1} = \sum_0^\infty \phi(-n-1)(-ix)^n = \sqrt{\left(\frac{\pi}{2}\right)}M(x),$$

which is (3.2).

4. We now suppose that $\phi(u)$ satisfies (2.1), so that $L(x)$ is even, and define ϕ_1, L_1, M_1 as in § 2. Then

$$|\psi_1(u)| = \left|\frac{\phi_1(u)}{\Gamma(1+u)}\right| = \sqrt{\left(\frac{2}{\pi}\right)}|\cos\tfrac{1}{2}u\pi\,\phi(-u-1)| = O(e^{P_1v+A_1|w|}),$$

where $$A_1 = B+\tfrac{1}{2}\pi < \pi, \qquad P_1 = Q, \tag{4.1}$$

for $v \geqslant -\delta$; and

$$|\phi_1(u)| = \sqrt{\left(\frac{2}{\pi}\right)}|\Gamma(1+u)\cos\tfrac{1}{2}u\pi\,\phi(-u-1)|$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left|\frac{\psi(-u-1)}{\sin\frac{1}{2}u\pi}\right| = O(e^{-Q_1v+B_1|w|}),$$

where $$B_1 = A-\tfrac{1}{2}\pi < \tfrac{1}{2}\pi, \qquad Q_1 = P, \tag{4.2}$$

for $v \leqslant -\delta$. Hence ϕ_1 satisfies conditions like those imposed upon ϕ, with A_1, B_1, P_1, Q_1 in the place of A, B, P, Q. Also

$$A = B_1+\tfrac{1}{2}\pi, \qquad P = Q_1, \qquad B = A_1-\tfrac{1}{2}\pi, \qquad P_1 = Q,$$

so that the relationship is fully symmetrical.

We thus obtain

THEOREM 2. *Suppose that $\phi(u)$ satisfies the conditions of Theorem 1, and that*

$$\phi(1) = \phi(3) = \phi(5) = \ldots = 0.$$

Then the series

$$L(x) = \sum_0^\infty \frac{\phi(2n)}{2n!}x^{2n}, \qquad M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_0^\infty (-1)^n\phi(-2n-1)x^{2n}$$

are convergent for $-e^{-P} < x < e^{-P}$ and $-e^{-Q} < x < e^{-Q}$ respectively. The functions $L(x)$ and $M(x)$ are regular for positive x; and

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty L(t)\cos xt\,dt = M(x), \qquad \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty M(t)\cos xt\,dt = L(x),$$

so that $L(x)$ and $M(x)$ are a pair of Fourier transforms.

There is, of course, a similar theorem for sine transforms. In this

$$\phi(0) = \phi(2) = \phi(4) = \ldots = 0,$$

$$\phi_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1+u)\sin\tfrac{1}{2}u\pi\,\phi(-u-1),$$

$$L(x) = \sum_0^\infty \frac{\phi(2n+1)}{(2n+1)!}x^{2n+1},$$

$$M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_0^\infty (-1)^{n+1}\phi(-2n-2)x^{2n+1}.$$

The inequality conditions on $\phi(u)$ are the same.

5. The self-reciprocal case. The condition that $L(x)$ should be its own reciprocal is

$$\phi_1(u) = \phi(u)$$

or

$$\phi(u) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1+u)\cos\tfrac{1}{2}u\pi\,\phi(-u-1). \tag{5.1}$$

If we write

$$\chi(u) = 2^{1-\frac{1}{2}u}\frac{\Gamma(u)}{\Gamma(\frac{1}{2}u)}\phi(-u) \tag{5.2}$$

then (5.1) reduces to

$$\chi(u) = \chi(1-u), \tag{5.3}$$

so that $\chi(u)$ is an even function of $u-\frac{1}{2}$. In this case

$$L(x) = M(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_0^\infty (-1)^n 2^{n-\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2})}{2n!}\chi(2n+1)x^{2n}$$

$$= \sum_0^\infty \frac{\chi(2n+1)}{n!}(-\tfrac{1}{2}x^2)^n.$$

When $\chi(u) = 1$, $L(x) = M(x) = e^{-\frac{1}{2}x^2}$.

The integral formula for $L(x)$ is

$$L(x) = -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{\pi}{\sin u\pi}\frac{\phi(u)}{\Gamma(1+u)}x^u\,du = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\Gamma(-u)\phi(u)x^u\,du$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(u)\phi(-u)x^{-u}\,du = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{\frac{1}{2}u-1}\Gamma(\tfrac{1}{2}u)\chi(u)x^u\,du,$$

where $0 < c < 1$. This is (apart from a factor 2) the formula of Hardy and Titchmarsh.* The conditions imposed upon $\chi(u)$ by our analysis are that (i) $\chi(u)$ is an integral function of u, and (ii) that

$$|\chi(u)| = O\{|\Gamma(\tfrac{1}{2}+\tfrac{1}{2}u)|e^{Kv+H|w|}\},$$

* Hardy and Titchmarsh (4), formula (1.46).

where $H < \frac{1}{2}\pi$ and $K = Q+\frac{1}{2}\log 2$, for $v \geqslant -\delta$. The series for $L(x)$ is convergent if $x^2 < e^{-2Q}$.

6. Examples.

(i) If we take

$$\phi(u) = 2^{-1-\frac{1}{2}u}\frac{\Gamma(-\frac{1}{2}u)}{\Gamma(-u)} = 2^{-\frac{1}{2}u}\cos\tfrac{1}{2}u\pi\frac{\Gamma(1+u)}{\Gamma(1+\frac{1}{2}u)},$$

then the conditions of Theorem 2 are satisfied, with $A = \frac{3}{4}\pi$, $B = \frac{1}{4}\pi$, and any values of P and Q. This is the case $L(x) = M(x) = e^{-\frac{1}{2}x^2}$.

(ii) The function $\eta(u)$, defined by the series

$$\eta(u) = 1^{-u}-3^{-u}+5^{-u}-...$$

and its analytic continuations, satisfies the equation

$$\eta(1-u) = 2^u\pi^{-u}\sin\tfrac{1}{2}u\pi\Gamma(u)\eta(u).$$

If we take $$\phi(u) = 2(\tfrac{1}{2}\pi)^{\frac{1}{2}u}\eta(-u),$$

it will be found that all the conditions of Theorem 2 are satisfied, with

$$A = \tfrac{1}{2}\pi, \qquad P = -\tfrac{1}{2}\log\tfrac{1}{2}\pi, \qquad B = 0, \qquad Q = -\tfrac{1}{2}\log\tfrac{1}{2}\pi.$$

Here also $\phi_1(u) = \phi(u)$, and

$$L(x) = \operatorname{sech} x\sqrt{(\tfrac{1}{2}\pi)}$$

is self-reciprocal.

(iii) If $$\phi(u) = \frac{\cos\frac{1}{2}u\pi}{u+1}$$

then $\phi(u)$ is integral and $\phi(1) = \phi(3) = ... = 0$. The conditions of Theorem 1 are not satisfied because*

$$\phi(iw) \sim \tfrac{1}{2}|w|^{-1}e^{\frac{1}{2}\pi|w|}.$$

The result of the theorem is, however, still true, the formulae (1.1) and (1.2) being

$$\int\limits_0^\infty \frac{\sin t}{t}\cos xt\,dt = \tfrac{1}{2}\pi, \qquad \int\limits_0^\infty \frac{\sin t}{t}\sin xt\,dt = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

These formulae hold for $-1 < x < 1$; but the values of the integrals are

$$0, \qquad \tfrac{1}{2}\log\frac{x+1}{x-1}$$

when $x > 1$, and $M(x)$ is not regular for all positive x.

* The values of A, P, B, Q in conditions (2) and (3) would be

$$A = \pi, \qquad P = -\infty, \qquad B = \tfrac{1}{2}\pi, \qquad Q = 0,$$

$P = -\infty$ meaning that (2) is satisfied for any P.

We could, of course, generalize the conditions of Theorem 1 so as to make it cover this and similar cases.

7. General transforms. There are analogues of Ramanujan's formulae in which $\cos x$ or $\sin x$ are replaced by a general 'Fourier kernel' $K(x)$. I confine myself to formal indications.

The Mellin transform $k(s)$ of $K(x)$ satisfies

$$k(s)k(1-s) = 1,$$

and we may write
$$k(s) = \frac{p(s)}{p(1-s)}.$$

I suppose that $p(s)$ is the reciprocal of an integral function. We start from the formula

$$L(x) = \sum_0^\infty (-1)^n \frac{\chi(n)}{p(n+1)} x^n = -\frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin u\pi} \frac{\chi(u)}{p(u+1)} x^u\, du,$$

where $-1 < \kappa < 0$, and argue as in § 3. We find that

$$\int_0^\infty K(xt)L(t)\, dt = M(x), \qquad \int_0^\infty K(xt)M(t)\, dt = L(x), \tag{7.1}$$

where

$$L(x) = \sum_0^\infty (-1)^n \frac{\chi(n)}{p(n+1)} x^n, \qquad M(x) = \sum_0^\infty (-1)^n \frac{\chi(-n-1)}{p(n+1)} x^n. \tag{7.2}$$

The condition for a self-reciprocal $L(x)$ is

$$\chi(u) = \chi(-u-1).$$

For example, when
$$K(u) = u^{\frac{1}{2}} J_\nu(u),$$

we have
$$k(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu+\frac{3}{4}-\frac{1}{2}s)};$$

and we may take
$$p(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4}).$$

The formulae (7.2) become

$$L(x) = \sum_0^\infty (-1)^n \frac{\chi(n)}{\Gamma(\frac{1}{2}n+\frac{1}{2}\nu+\frac{3}{4})} x^n,$$

$$M(x) = \sum_0^\infty (-1)^n \frac{\chi(-n-1)}{\Gamma(\frac{1}{2}n+\frac{1}{2}\nu+\frac{3}{4})} x^n.$$

When
$$K(x) = \frac{2}{\pi} \frac{1}{1-x^2}$$

we have
$$k(s) = \cot \tfrac{1}{2}s\pi,$$

and we may take
$$p(s) = \operatorname{cosec} \tfrac{1}{2}s\pi.$$

The formulae are then

$$L(x) = \sum_0^\infty (-1)^n \chi(2n) x^{2n}, \qquad M(x) = \sum_0^\infty (-1)^n \chi(-2n-1) x^{2n}.$$

Thus $\chi(u) = 1$ gives the self-reciprocal function

$$\frac{1}{1+x^2}.$$

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ON THE DIFFERENCE BETWEEN CONSECUTIVE PRIMES

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1. LET p_n denote the nth prime, and $\pi(x)$ the number of primes p not exceeding x. The existence of an absolute constant $\theta < 1$ such that

$$\pi(x+x^\theta)-\pi(x) \sim \frac{x^\theta}{\log x} \tag{1}$$

when $x \to \infty$, and therefore

$$p_{n+1}-p_n = O(p_n^\theta) \tag{2}$$

when $n \to \infty$, was first proved by Hoheisel.† His proof was based on two propositions concerning the zeros of the Riemann zeta-function $\zeta(s) = \zeta(\sigma+ti)$:

(i) Littlewood's theorem‡ that $\zeta(s)$ has no zeros in a domain of the type

$$\sigma > 1-A\frac{\log\log t}{\log t}, \qquad t > t_0,$$

where $A > 0$, $t_0 > 3$;

(ii) Carlson's theorem, or rather a refinement of it,§ namely that

$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)}\log^6 T) \tag{3}$$

uniformly for $\frac{1}{2} < \frac{1}{2}+\delta \leqslant \sigma \leqslant 1$ as $T \to \infty$, where $N(\sigma, T)$ is the number of zeros $\rho = \beta+\gamma i$ of $\zeta(s)$ with $\beta \geqslant \sigma$, $0 < \gamma \leqslant T$.

Hoheisel proved (1) and (2) with $\theta = \frac{32999}{33000}$, and Heilbronn|| reduced this to $\frac{249}{250}$ by increasing the numerical value of A in (i). It was pointed out by Hoheisel in his original paper that the value $\theta = \frac{3}{4}+\epsilon$ (where ϵ is an arbitrarily small positive number) would follow from his analysis if A could be replaced by an $A(t)$ tending to infinity with t. This advance has now been made by Tchudakoff with the aid of Vinogradoff's results on the estimation of trigonometrical sums.††

The aim of this paper is to reduce the index θ still further by a reconsideration of the exponent of T in (3). The main result (Theorem 4) is that, if

$$\zeta(\tfrac{1}{2}+ti) = O(t^c) \tag{4}$$

† Hoheisel (8). See the list of references at the end.

‡ Landau (12), ii, Satz 397; Titchmarsh (15), Theorem 13.

§ Hoheisel (8), § 2. || Heilbronn (7). †† Tchudakoff (1), (2).

as $t \to \infty$, where c is a positive (absolute) constant, then (1) and (2) are true with

$$\theta = \frac{1+4c}{2+4c}+\epsilon.$$

Thus even the classical value $c = \frac{1}{4}+\epsilon$ reduces θ from $\frac{3}{4}+\epsilon$ to $\frac{2}{3}+\epsilon$. The Hardy-Littlewood value† $c = \frac{1}{6}+\epsilon$ gives $\theta = \frac{5}{8}+\epsilon$, while the best published value‡ $c = \frac{229}{1392}$ gives $\theta = \frac{577}{925}+\epsilon$. Moreover, *if the Lindelöf hypothesis is true* (that is to say, if (4) holds with an arbitrarily small c), then (1) and (2) are true with $\theta = \frac{1}{2}+\epsilon$. This may be compared with Cramér's theorem§ that, *if the Riemann hypothesis is true*, then

$$p_{n+1}-p_n = O(p_n^{\frac{1}{2}}\log p_n).$$

The standard proofs of theorems of the type of (3) are based on the estimation of a certain integral (the integral in Theorem 2 below). In Heilbronn's proof of (1) and (2) this integral is introduced directly into the arithmetical problem without explicit mention of $N(\sigma, T)$, and his method has the advantage of being applicable to other problems. In spite of this we shall follow Hoheisel's line of argument, partly because $N(\sigma, T)$ is of interest in itself and partly because a new difficulty appears in Heilbronn's method with the reduction of the index θ below $\frac{3}{4}$.

The inequality for $N(\sigma, T)$ which we actually prove (in Theorem 3) for application to (1) and (2) is of interest only in the neighbourhood of $\sigma = 1$. In Theorem 5 we indicate the proof of a result which, though less precise in this particular region and less useful for the arithmetical application, is perhaps more interesting in itself in that it supersedes existing theorems over the whole range $\frac{1}{2} < \sigma < 1$. Taken together the two theorems give

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ as $T \to \infty$, where

$$\lambda(\sigma) = \min(1+2\sigma,\ 2+4c),$$

† Landau (**12**), ii, Satz 414; Titchmarsh (**15**), Theorem 15.

‡ Phillips (**13**). A better value $c = \frac{19}{116}$ $(= \frac{228}{1392})$ has been obtained by Prof. Titchmarsh in an unpublished manuscript. This gives $\theta = \frac{48}{77}+\epsilon$.

§ Cramér (**3**); see also (**4**). Another proof of this result may be constructed by performing the operation $\Delta_h^{(2)}$, where $\Delta_h^{(2)} f(x) = f(x+2h)-2f(x+h)+f(x)$, on the explicit formula for $\psi_1(x)$ [Theorem 28 of my tract (**10**)], using the inequality

$$\left|\frac{\Delta_h^{(2)} x^{\rho+1}}{\rho(\rho+1)}\right| \leqslant \min\left(h^2 x^{\Theta-1}, \frac{4(2x)^{\Theta+1}}{\gamma^2}\right) \quad (1 \leqslant h \leqslant x)$$

(where Θ is the upper bound of $\Re\rho$), and taking $h = Cx^{\Theta}\log x$ with a sufficiently large absolute constant C.

c being a number for which (4) is true. The index $(1+2\sigma)(1-\sigma)$ of Theorem 5 is an advance on Titchmarsh's index†

$$1-(2\sigma-1)/(3-2\sigma) = 4(1-\sigma)/(3-2\sigma),$$

since $$\frac{4}{3-2\sigma}-(1+2\sigma) = \frac{(2\sigma-1)^2}{3-2\sigma} > 0 \quad (\tfrac{1}{2} < \sigma \leqslant 1).$$

The principal weapon is a convexity theorem for integrals.‡

2. Theorem 1. *Suppose* (i) *that $\zeta(s)$ has no zeros in the domain*

$$\sigma > 1-A\frac{\log\log t}{\log t} \qquad (A > 0;\, t > t_0 > 3),$$

and (ii) *that* $$N(\sigma,T) = O(T^{b(1-\sigma)}\log^B T) \tag{5}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ as $T \to \infty$, where $b > 0$, $B \geqslant 0$. Then (1) *and* (2) *are true for any fixed θ satisfying*

$$1-\frac{1}{b+A^{-1}B} < \theta < 1. \tag{6}$$

This is essentially the content of Hoheisel's main theorem. For completeness we reproduce the proof (simplified in detail).

By a known formula§ we have

$$\psi(x) = x-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}+O\left(\frac{x}{T}\log^2 x\right)$$

uniformly for $3 \leqslant T \leqslant x$ as $x \to \infty$, where $\psi(x) = \sum_{p^m \leqslant x}\log p$, and $\rho = \beta+\gamma i$ is a typical complex zero of $\zeta(s)$. Hence

$$\psi(x+h)-\psi(x) = h-\sum_{|\gamma|<T}\frac{(x+h)^\rho-x^\rho}{\rho}+O\left(\frac{x}{T}\log^2 x\right)$$

where O's are uniform for $3 \leqslant T \leqslant x$, $0 < h \leqslant x$, as $x \to \infty$. Since

$$\left|\frac{(x+h)^\rho-x^\rho}{\rho}\right| = \left|\int_x^{x+h} u^{\rho-1}\,du\right| \leqslant \int_x^{x+h} u^{\beta-1}\,du \leqslant hx^{\beta-1},$$

this implies that

$$\frac{\psi(x+h)-\psi(x)}{h} = 1+O\Big(\sum_{|\gamma|\leqslant T}x^{\beta-1}\Big)+O\left(\frac{x}{Th}\log^2 x\right). \tag{7}$$

† Titchmarsh (**14**). ‡ Hardy, Ingham, and Pólya (**5**), Theorem 7.
§ Landau (**11**), Satz 1.

Now we have†

$$\sum_{|\gamma|\leqslant T}(x^{\beta-1}-x^{-1}) = \sum_{|\gamma|\leqslant T}\int\limits_0^{\beta} x^{\sigma-1}\log x\,d\sigma = \int\limits_0^1 \sum_{\substack{|\gamma|\leqslant T\\ \beta\geqslant\sigma}} x^{\sigma-1}\log x\,d\sigma,$$

or

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = 2x^{-1}N(0,T)+2\int\limits_0^1 N(\sigma,T)x^{\sigma-1}\log x\,d\sigma. \tag{8}$$

Since $\zeta(s)$ has only a finite number of zeros $\rho = \beta+\gamma i$ with $\frac{1}{2} \leqslant \beta < 1$, $|\gamma| \leqslant t_0$ and none with $\beta \geqslant 1$, it follows from hypothesis (i) that we can find a $T_0 > 3$ so that $N(\sigma, T) = 0$ for $T \geqslant T_0$, $\sigma > 1-\eta(T)$, where $\eta(T) = A(\log\log T)/(\log T)$. Further, since $N(\frac{1}{2}, T) \neq o(T)$, the case $\sigma = \frac{1}{2}$ of hypothesis (ii) shows that $b \geqslant 2$. Also $N(\sigma, T) \leqslant 2N(\frac{1}{2}, T)$ for $\sigma \leqslant \frac{1}{2}$, so that (5) holds uniformly for $0 \leqslant \sigma \leqslant 1$. Hence by (8), since $N(0,T) = O(T\log T)$,

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = O(x^{-1}T\log T)+O\left(\int\limits_0^{1-\eta(T)} \left(\frac{T^b}{x}\right)^{1-\sigma} \log^B T\log x\,d\sigma\right)$$

uniformly for $T_0 \leqslant T \leqslant x$ as $x \to \infty$.

Take $T = x^{\alpha}$, where α is a constant satisfying $0 < \alpha < b^{-1}$ ($\leqslant \frac{1}{2}$). Then

$$\begin{aligned}\sum_{|\gamma|\leqslant T} x^{\beta-1} &= O(x^{\alpha-1}\log x)+O(x^{(\alpha b-1)\eta(x^{\alpha})}\log^B x)\\ &= O(x^{\alpha-1}\log x)+O(e^{(b-\alpha^{-1})A\log(\alpha\log x)}\log^B x) = O\big((\log x)^{-\delta}\big),\end{aligned} \tag{9}$$

where $\delta = (\alpha^{-1}-b)A-B$. Choose α so that $\alpha^{-1} > b+A^{-1}B$ ($\geqslant b$). Then $\delta > 0$, so that by (9) and (7) (with $T = x^{\alpha}$)

$$\psi(x+h)-\psi(x) \sim h$$

when $x \to \infty$, if $h = x^{\theta}$ and θ is a constant satisfying

$$1 > \theta > 1-\alpha \ (> \tfrac{1}{2}).$$

This implies (1), and therefore (2), since (when $h = x^{\theta}$)

$$\begin{aligned}\psi(x+h)-\psi(x) &= \sum_{x<p\leqslant x+h}\log p+O\left(\sum_{p^2\leqslant x+h}\log p\left[\frac{\log(x+h)}{\log p}\right]\right)\\ &= \sum_{x<p\leqslant x+h}\{\log x+O(1)\}+O\left(\sum_{p^2\leqslant 2x}\log 2x\right)\\ &= \{\pi(x+h)-\pi(x)\}\{\log x+O(1)\}+O(x^{\frac{1}{2}}\log x).\end{aligned}$$

† Or, using the Stieltjes integral,

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = -2\int\limits_0^{1+0} x^{\sigma-1}d_{\sigma}N(\sigma,T) = 2x^{-1}N(0,T)+2\int\limits_0^1 N(\sigma,T)d_{\sigma}(x^{\sigma-1}).$$

The conditions on α and θ imply (6), and for any given θ satisfying (6) an appropriate α can be found.

3. THEOREM 2. *Let*

$$f_X(s) = \zeta(s) \sum_{n<X} \mu(n)n^{-s}-1 = \zeta(s)M_X(s)-1,$$

where $\mu(n)$ is Möbius's function. Then, if c is an absolute constant for which (4) *is true,*

$$\int_1^T |f_X(\sigma+ti)|^2\,dt < C\frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}}(T+X)\log^4(T+X) \tag{10}$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$, $X > 1$, where C is a positive absolute constant.

We may suppose that $X \geqslant 2$, since $f_X(s) = f_2(s)$ for $1 < X < 2$. We subject T in the first instance only to the restriction $T \geqslant 0$. The absolute constant c of (4) is necessarily positive, and we may suppose that $c < \frac{1}{2}$, since (4) is certainly true for some $c < \frac{1}{2}$. The symbols $C_1, C_2,...$ denote positive absolute constants.

For $\sigma > 1$ we have

$$f_X(s) = \sum_{n \geqslant X} a_X(n)n^{-s},$$

where

$$a_X(n) = \sum_{\substack{d|n \\ d<X}} \mu(d), \tag{11}$$

so that $a_X(1) = 1$, $a_X(n) = 0$ for $1 < n < X$, and $|a_X(n)| \leqslant d(n)$ for all n. Hence, if $0 < \delta < 1$,

$$\int_0^T |f_X(1+\delta+ti)|^2\,dt = \sum_{m,n\geqslant X} \frac{a_X(m)a_X(n)}{(mn)^{1+\delta}} \int_0^T \left(\frac{m}{n}\right)^{ti} dt = \sum_{m=n} + 2\Re \sum_{m<n}$$

$$\leqslant T\sum_{n\geqslant X}\frac{d^2(n)}{n^{2+2\delta}} + 4\sum_{n>m\geqslant X}\frac{d(m)d(n)}{(mn)^{1+\delta}\log(n/m)}. \tag{12}$$

These sums are easily estimated by means of the known inequalities†

$$\sum_{n\leqslant x} d^2(n) < C_1 x\log^3 x \quad (x \geqslant 2), \tag{13}$$

$$\sum_{m<n\leqslant x} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)} < C_2 x\log^3 x \quad (x > 1). \tag{14}$$

† For (14) see Ingham (9), 296, Lemmas B2 and B3. (13) is included in an asymptotic formula stated by S. Ramanujan and proved analytically by B. M. Wilson, but an elementary proof of (13) is suggested by Heilbronn (7) (cf. 413, Hilfssatz 20). It may be remarked in passing that the result and proof of Lemma B2 of my paper (9) remain valid even in the case $k = 0$ if we replace $\sum_{d|k}$ by $\sum_{\substack{d|k \\ d\leqslant\sqrt{x}}}$; this provides another elementary proof of (13).

We deduce, in fact, from (13) that, for $0 < \xi < 3$,

$$\sum_{n \geqslant X} \frac{d^2(n)}{n^{1+\xi}} = \sum_{n \geqslant X} d^2(n) \int_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx = \int_X^\infty \frac{1+\xi}{x^{2+\xi}} \sum_{X \leqslant n \leqslant x} d^2(n)\,dx$$

$$< \int_X^\infty \frac{(1+\xi)C_1 \log^3 x}{x^{1+\xi}}\,dx < \frac{C_3}{\xi X^\xi}\left(\frac{1}{\xi}+\log X\right)^3 \tag{15}$$

by the substitution $x = Xy^{1/\xi}$; and from (14) (since $1 < \log\lambda+\lambda^{-1} < \log\lambda+\lambda^{-\frac{1}{2}}$ for $\lambda > 1$) that

$$\sum_{n>m\geqslant X} \frac{d(m)d(n)}{(mn)^{1+\xi}\log(n/m)} < \sum_{n>m\geqslant X} \frac{d(m)d(n)}{(mn)^{1+\xi}} + \sum_{n>m\geqslant X} \frac{d(m)d(n)}{m^\xi n^{1+\xi}(mn)^{\frac{1}{2}}\log(n/m)}$$

$$< \left(\sum_{n=1}^\infty \frac{d(n)}{n^{1+\xi}}\right)^2 + \sum_{n>m\geqslant 1} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)} \int_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx$$

$$= \zeta^4(1+\xi) + \int_1^\infty \frac{1+\xi}{x^{2+\xi}} \sum_{m<n\leqslant x} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}\,dx$$

$$< \zeta^4(1+\xi) + \int_1^\infty \frac{(1+\xi)C_2 \log^3 x}{x^{1+\xi}}\,dx < \frac{C_4}{\xi^4}. \tag{16}$$

Hence by (12) (since $(\log X)^3/X^{2\delta} < \delta^{-3}$)

$$\int_0^T |f_X(1+\delta+ti)|^2\,dt < C_5\left(\frac{T}{X}+1\right)\delta^{-4}. \tag{17}$$

For $\sigma = \frac{1}{2}$ we use the inequalities

$$|f_X|^2 \leqslant 2(|\zeta|^2|M_X|^2+1),$$

$$\int_0^T |M_X(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<X} \frac{\mu^2(n)}{n} + 4\sum_{m<n<X} \frac{|\mu(m)\mu(n)|}{(mn)^{\frac{1}{2}}\log(n/m)}$$

$$\leqslant T\sum_{n<X} \frac{1}{n} + 4\sum_{m<n<X}\left(\frac{1}{(mn)^{\frac{1}{2}}}+\frac{1}{n-m}\right) < C_6(T+X)\log X$$

(since $1/\log\lambda < \lambda/(\lambda-1) < 1+\lambda^{\frac{1}{2}}/(\lambda-1)$ for $\lambda > 1$), and deduce by (4) that

$$\int_0^T |f_X(\tfrac{1}{2}+ti)|^2\,dt < C_7 T^{2c}(T+X)\log X, \tag{18}$$

the inequality holding down to $T = 0$ since the integral is at most $C_8 X.T \leqslant C_8 T^{2c} X$ for $0 \leqslant T \leqslant 1$.

From (17) and (18) we shall deduce an inequality valid for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$ by means of a convexity theorem. Write

$$I_\sigma(T) = \int_0^T |f_X(\sigma+ti)|^2\,dt, \qquad J_\sigma = \int_{-\infty}^{\infty} |\phi(\sigma+ti)|^2\,dt,$$

where $$\phi(s) = \phi_{X,\tau}(s) = \frac{s-1}{s\cos(s/2\tau)} f_X(s) \quad (\tau > 3/\pi).$$

In the strip $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$, $\phi(s)$ is regular and satisfies

$$|\phi(s)|^2 \leqslant C_9 e^{-|t|/\tau} |f_X(s)|^2 \tag{19}$$

and is therefore certainly bounded (for fixed X and τ). Further, for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$, $\sigma \neq 1$,

$$J_\sigma \leqslant 2\int_0^\infty C_9 e^{-t/\tau} |f_X(\sigma+ti)|^2\,dt = 2C_9 \int_0^\infty e^{-u} I_\sigma(\tau u)\,du,$$

by partial integration and the substitution $t = \tau u$; whence, by (17) and (18),

$$J_{1+\delta} < C_{10} \int_0^\infty e^{-u}\left(\frac{\tau u}{X}+1\right)\delta^{-4}\,du < C_{11}\left(\frac{\tau}{X}+1\right)\delta^{-4},$$

$$J_{\frac{1}{2}} < C_{12} \int_0^\infty e^{-u}(\tau u)^{2c}(\tau u+X)\log X\,du < C_{13}\tau^{2c}(\tau+X)\log X.$$

It follows† that, for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$,

$$J_\sigma < \left\{C_{11}\left(\frac{\tau}{X}+1\right)\delta^{-4}\right\}^{\frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta}} \{C_{13}\tau^{2c}(\tau+X)\log X\}^{\frac{1+\delta-\sigma}{\frac{1}{2}+\delta}}. \tag{20}$$

Now $$|\phi(s)|^2 \geqslant C_{14} e^{-t/\tau} |f_X(s)|^2 \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1+\delta,\ t \geqslant 1).$$

Hence (20) implies that, for $T > 1$, $\frac{1}{2} \leqslant \sigma \leqslant 1$,

$$C_{14} e^{-T/\tau} \int_1^T |f_X(\sigma+ti)|^2\,dt$$
$$< X^{\frac{1-2\sigma}{1+2\delta}} \tau^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} (\tau+X)\max(C_{11}\delta^{-4}, C_{13}\log X)$$

(on simplification of the right-hand side). Taking $\tau = C_{15}T$, $\delta = C_{16}/\log(T+X)$, we deduce the theorem, since

$$X^{\frac{1-2\sigma}{1+2\delta}} \leqslant X^{-(1-2\delta)(2\sigma-1)} \leqslant X^{-(2\sigma-1)+2\delta} < e^{2C_{16}} X^{-(2\sigma-1)},$$

$$T^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} \leqslant T^{4c(1+\delta-\sigma)} \leqslant T^{4c(1-\sigma)+2\delta} < e^{2C_{16}} T^{4c(1-\sigma)}.$$

† Hardy, Ingham, and Pólya (5), Theorem 7.

THEOREM 3. *If* (4) *is true, then*

$$N(\sigma, T) = O(T^{2(1+2c)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ *when* $T \to \infty$.

This may be deduced from Theorem 2 by a familiar argument. We have, in the notation of Theorem 2,

$$1-f_X^2 = \zeta M_X(2-\zeta M_X) = \zeta g = h,$$

where $g(s) = g_X(s)$ and $h(s) = h_X(s)$ are regular except at $s = 1$; and

$$\log|h| \leqslant \log(1+|f_X|^2) \leqslant |f_X|^2. \tag{21}$$

Also, for $\sigma \geqslant 2$,

$$|f_X|^2 \leqslant \Big(\sum_{n \geqslant X} \frac{d(n)}{n^2}\Big)^2 < \frac{1}{2X} < \frac{1}{2},$$

if $X > C_{17} > 1$; whence

$$\left.\begin{aligned} \Re h &> \tfrac{1}{2} \\ -\log|h| \leqslant -\log(1-|f_X|^2) &\leqslant 2|f_X|^2 < X^{-1} \end{aligned}\right\} \quad (\sigma \geqslant 2; X > C_{17}). \qquad \begin{matrix}(22)\\(23)\end{matrix}$$

Take $X > C_{17}$, $T > 4$, and choose T_1 and T_2 so that $3 < T_1 < 4$, $T < T_2 < T+1$, and $h(s)$ has no zeros on either of the segments $t = T_1$ or $t = T_2$ ($\frac{1}{2} \leqslant \sigma \leqslant 2$). Then, writing

$$N_\zeta(\sigma; T_1, T_2) = N_\zeta(\sigma, T_2) - N_\zeta(\sigma, T_1),$$

where the suffix ζ refers to the function $\zeta(s)$, and extending the notation to $g(s)$ and $h(s)$, we have, applying a theorem of Littlewood† and taking account of (22),

$$\int_\sigma^2 N_h(\sigma; T_1, T_2)\, d\sigma = \int_{T_1}^{T_2} (\log|h(\sigma_0+ti)| - \log|h(2+ti)|)\, dt + $$
$$+ \int_{\sigma_0}^2 (\arg h(\sigma+T_2 i) - \arg h(\sigma+T_1 i))\, d\sigma \tag{24}$$

for $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$, where $\arg h(s)$ is 0 when $s = 2$ and varies continuously along the lines $\sigma = 2$, $t = T_1$, $t = T_2$. By (21), Theorem 2, and (23), the first integral on the right is less than

$$C(T+1)^{4c(1-\sigma_0)} X^{1-2\sigma_0}(T+1+X)\log^4(T+1+X) + TX^{-1}.$$

In the second integral

$$|\arg h(\sigma+T_r i)| \leqslant (m_r+1)\pi \quad (r = 1, 2),$$

where m_r is the number of points of the segment $t = T_r$, $\sigma_0 < \sigma < 2$, at which $\Re h(s) = 0$; for $\arg h(s)$ cannot vary by more than π on any of the m_r+1 pieces into which these points divide the broken line

† Titchmarsh (**15**), 3.52, 3.53; or (**16**), 3.8.

$(2, 2+T_r i,\ \sigma_0+T_r i)$, since $\Re h(s) \neq 0$ on the vertical part by (22). But m_r is the number of zeros of the function

$$H_r(s) = \tfrac{1}{2}\{h(s+T_r i)+h(s-T_r i)\}$$

on the segment $t = 0$, $\sigma_0 < \sigma < 2$, and therefore cannot exceed the number of zeros of $H_r(s)$ in the circle $|s-2| \leqslant \frac{3}{2}$. Hence, since $H_r(s)$ is regular for $|s-2| \leqslant \frac{7}{4}$, we have†

$$\left(\frac{7}{6}\right)^{m_r} \leqslant \max_{|s-2|\leqslant\frac{7}{4}} \left|\frac{H_r(s)}{H_r(2)}\right| \leqslant \max_{\substack{\sigma\geqslant\frac{1}{4}\\ 1\leqslant t\leqslant T+3}} \frac{|h(s)|}{\Re h(2+T_r i)} < (T+X)^{C_{18}}$$

by (22) and the definition of $h(s)$. The second integral on the right of (24) is therefore less than $C_{19}\log(T+X)$. Collecting these results, we obtain

$$\int\limits_{\sigma_0}^{2} N_h(\sigma; T_1, T_2)\, d\sigma < C_{20}\, T^{4c(1-\sigma_0)}(TX^{1-2\sigma_0}+X^{2(1-\sigma_0)})\log^4(T+X),$$

since $TX^{-1} \leqslant TX^{1-2\sigma_0}$ and $\log(T+X) \leqslant X^{2(1-\sigma_0)}\log(T+X)$.

On the other hand, since $N_h = N_\zeta+N_g \geqslant N_\zeta$,

$$\int\limits_{\sigma_0}^{2} N_h(\sigma; T_1, T_2)\, d\sigma \geqslant \int\limits_{\sigma_0}^{\sigma_0+\delta} N_\zeta(\sigma; T_1, T_2)\, d\sigma \geqslant \delta N_\zeta(\sigma_0+\delta; T_1, T_2),$$

if $0 < \delta < 1$. Writing σ for $\sigma_0+\delta$, we deduce, since

$$N_\zeta(\sigma, T) < N_\zeta(\sigma; T_1, T_2)+C_{21},$$

that

$$N_\zeta(\sigma, T) < C_{22}\,\delta^{-1}T^{4c(1-\sigma+\delta)}(TX^{1-2\sigma+2\delta}+X^{2(1-\sigma+\delta)})\log^4(T+X)$$
$$(\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 1).$$

But (since $T > 4$)

$$N_\zeta(\sigma, T) < C_{23}\, T\log T \leqslant C_{23}\, T^{2(1-\sigma+\delta)}\log T \quad (\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}+\delta).$$

The theorem follows from these inequalities if we take

$$X = T > \max(C_{17}, 4), \qquad \delta = 1/\log T.$$

4. Theorem 4. *If* (4) *is true, then* (1) *and* (2) *are true for any fixed* θ *satisfying*

$$\frac{1+4c}{2+4c} < \theta < 1.$$

† By a well-known corollary of Jensen's formula. For a direct proof of this particular result see, for example, Ingham (**10**), 49, Theorem D.

Tchudakoff has proved† that $\zeta(s)$ has no zeros in a domain of the type

$$1-\frac{1}{(\log t)^{a}}, \quad t>t_1,$$

where $a<1$, $t_1>2$. The condition (i) of Theorem 1 is therefore satisfied with an arbitrarily large A and an appropriate t_0.

By Theorem 3 the condition (ii) of Theorem 1 is satisfied with $b=2+4c$ and $B=5$.

Theorem 4 therefore follows from Theorem 1, since

$$\lim_{A\to\infty}\left(1-\frac{1}{b+A^{-1}B}\right)=1-\frac{1}{b}=\frac{1+4c}{2+4c}.$$

5. Theorem 3 is specially designed for application to Theorem 4. The result itself is of no interest over the range

$$\tfrac{1}{2}\leqslant\sigma<(1+4c)/(2+4c)$$

where the exponent of T is greater than 1. We shall now give a result which is non-trivial over the whole range $\frac{1}{2}<\sigma<1$. It is better than Theorem 3 for $\frac{1}{2}<\sigma<\frac{1}{2}+2c$, but worse for $\frac{1}{2}+2c<\sigma<1$. Where the argument is similar to that of § 3 some of the details will be omitted.

THEOREM 5. *We have*

$$N(\sigma,T)=O(T^{(1+2\sigma)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2}\leqslant\sigma\leqslant 1$ *when* $T\to\infty$.

We need a new estimate of the integral of Theorem 2. We consider first the corresponding integral with $f_X(s)$ replaced by

$$f_{X,Y}(s)=\zeta(s)\sum_{n<X}\mu(n)n^{-s}-\sum_{n<Y}a_X(n)n^{-s}=\zeta(s)M_X(s)-A_{X,Y}(s),$$

say, where $2\leqslant X\leqslant Y$, and the coefficients $a_X(n)$ are defined by (11). Suppose that $T\geqslant 0$.

For $\sigma>1$ we have

$$f_{X,Y}(s)=\sum_{n\geqslant Y}a_X(n)n^{-s},$$

whence, as in § 3,

$$\int\limits_0^T|f_{X,Y}(1+\delta+ti)|^2\,dt<C_5\left(\frac{T}{Y}+1\right)\delta^{-4}\quad(0<\delta<1). \tag{25}$$

† Tchudakoff (**1**). By using the full force of this result we could replace $x^{\theta}=x^{(1+4c)/(2+4c)+\epsilon}=x^{\theta_0+\epsilon}$ in (1) by $x^{\theta_0}e^{(\log x)^{a'}}$ $(a<a'<1)$, and indeed by something a little better. [Cf. Tchudakoff (**2**).]

For $\sigma = \frac{1}{2}$ we use the inequalities†

$$\int_0^T |f_{X,Y}|^2\,dt \leqslant \int_0^T 2(|\zeta M_X|^2+|A_{X,Y}|^2)\,dt$$

$$\leqslant 2\Big(\int_0^T |\zeta|^4\,dt \int_0^T |M_X^2|^2\,dt\Big)^{\frac{1}{2}}+2\int_0^T |A_{X,Y}|^2\,dt;$$

$$\int_0^T |\zeta(\tfrac{1}{2}+ti)|^4\,dt < C_{24}\,T\log^4(T+2), \tag{26}$$

$$\int_0^T |A_{X,Y}(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<Y}\frac{d^2(n)}{n}+4\sum_{m<n<Y}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}$$

$$< C_{25}\,T\log^4 Y+4C_2\,Y\log^3 Y, \tag{27}$$

$$\int_0^T |M_X^2(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<X^2}\frac{d^2(n)}{n}+4\sum_{m<n<X^2}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}$$

$$< C_{25}\,T\log^4 X^2+4C_2 X^2\log^3 X^2, \tag{28}$$

and obtain,

$$\int_0^T |f_{X,Y}(\tfrac{1}{2}+ti)|^2\,dt < C_{26}(T+T^{\frac{1}{2}}X+Y)\log^4(T+Y)$$

$$< C_{27}(T+T^{\frac{1}{2}}X+Y)^{1+2\delta}\delta^{-4}. \tag{29}$$

Using a convexity argument as in § 3 and taking

$$\delta = C_{28}/\log(T+Y) < 2C_{28}/\log(T+T^{\frac{1}{2}}X+Y),$$

we deduce from (25) and (29) that, for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$,

$$\int_1^T |f_{X,Y}(\sigma+ti)|^2\,dt < C_{29}\Big(\frac{T}{Y}+1\Big)^{2\sigma-1}(T+T^{\frac{1}{2}}X+Y)^{2(1-\sigma)}\log^4(T+Y). \tag{30}$$

Now

$$f_X(s) = f_{X,Y}(s)+\sum_{X\leqslant n<Y} a_X(n)n^{-s} = f_{X,Y}(s)+A^*_{X,Y}(s), \tag{31}$$

say, and (for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$)‡

$$\int_1^T |A^*_{X,Y}(\sigma+ti)|^2\,dt \leqslant T\sum_{X\leqslant n<Y}\frac{d^2(n)}{n^{2\sigma}}+4\sum_{X\leqslant m<n<Y}\frac{d(m)d(n)}{(mn)^{\sigma}\log(n/m)}$$

$$< C_{30}(TX^{1-2\sigma}+Y^{2(1-\sigma)})\log^4 Y.$$

† (26) is a well-known result of Hardy and Littlewood (**6**), Theorem D (3.12); see also Titchmarsh (**17**). (27) and (28) follow from (13) and (14).

‡ The inequalities used here may be deduced from (13) and (14) in much the same way as (15) and (16) were deduced. Alternatively, we can deduce them directly from (15) and (16) by using the inequalities $n^{-2\sigma} \leqslant X^{1-2\sigma}Y^{\xi}n^{-1-\xi}$, $(mn)^{-\sigma} \leqslant Y^{2(1-\sigma+\xi)}(mn)^{-1-\xi}$ $(\xi > 0)$, and taking $\xi = 1/\log Y$.

Combining this with (30), we obtain, by (31),

$$\int_1^T |f_X(\sigma+ti)|^2\,dt$$
$$< C_{31}\left\{\left(\frac{T}{Y}+1\right)^{2\sigma-1}(T+T^{\frac{1}{2}}X+Y)^{2(1-\sigma)}+TX^{1-2\sigma}\right\}\log^4(T+Y).$$

By considering separately the ranges $Y \geqslant T$ and $Y \leqslant T$, we see at once that (for given T, X, σ) the right-hand side of this is of lowest order when $Y = T$, and this choice gives

$$\int_1^T |f_X(\sigma+ti)|^2\,dt < C_{32}(T^{2(1-\sigma)}+T^{1-\sigma}X^{2(1-\sigma)}+TX^{1-2\sigma})\log^4 T$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $2 \leqslant X \leqslant T$.

Arguing now as in the deduction of Theorem 3 from Theorem 2, and taking $X = C_{33}\,T^{\sigma}$, we obtain Theorem 5.

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INTEGRALDARSTELLUNGEN HYPERGEOMETRISCHER FUNKTIONEN

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1. In einer früheren Arbeit† wurde gezeigt, dass die verschiedenen Eulerschen Integrale, welche die Gausssche hypergeometrische Reihe darstellen, und ebenso die die Kummersche konfluente hypergeometrische Reihe darstellenden Laplaceschen Integrale als Sonderfälle gewisser allgemeinerer für diese Reihen geltender Funktionalbeziehungen aufgefasst werden können. Im Falle der Gaussschen Reihe lautet z.B. diese Funktionalbeziehung‡

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}{}_2F_1(a,b;\lambda;xt)\,dt \qquad (|x|<1;\ 0<\Re(\lambda)<\Re(c)). \tag{1.1} $$

Man wird auf (1.1) geführt, wenn man die Frage aufwirft, in welcher Weise die Identität der beiden durch die Annahmen $\lambda = b$ und $\lambda = a$ aus dieser Integralformel entspringenden Eulerschen Integrale

$$ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \quad (0<\Re(b)<\Re(c)) \tag{1.2} $$

und

$$ \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}\,dt \quad (0<\Re(a)<\Re(c)) \tag{1.3} $$

gezeigt werden kann. Hierzu bieten sich zwei Möglichkeiten. Der erste Weg—er wurde auf Grund einer Andeutung B. Riemanns§ von W. Wirtinger|| beschritten—besteht darin, *beide* soeben angeschriebenen Integrale auf ein und dasselbe *mehrfache* Integral zurückzuführen, wodurch ja ihre Identität evident wird. Der zweite—vom Verfasser†† eingeschlagene—Weg besteht darin, dass man versucht eine noch einen willkürlichen Parameter enthaltende allgemeinere Integralformel zu konstruieren, als deren Sonderfall sich dann die beiden Eulerschen Integrale entpuppen. Auf diese

† 2.

‡ 2 Gleichung (2.1).

§ 3 Seite 62 ff.

|| 6.

†† 2.

Weise ergibt sich (1.1). Vom Standpunkt dieser letzteren Methode aus gesehen beruht die Gleichheit der beiden Funktionen (1.2) und (1.3) darauf, dass die rechte Seite von (1.1) von dem willkürlichen Parameter λ *unabhängig* ist.

(1.1) und die zu ähnlichem Zwecke in der Theorie der Kummerschen Reihe heranzuziehende Integralformel† hängen mit gewissen Formeln der Operatorenrechnung zusammen.

Ganz ähnliche Probleme treten nun in der Theorie der verallgemeinerten hypergeometrischen Reihen

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots, a_p; \\ c_1,\ldots, c_q; \end{matrix} x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(c_1)_n\ldots(c_q)_n}\frac{x^n}{n!} \quad (p \leqslant q+1) \tag{1.4}$$

auf.‡ Nehmen wir zum Beispiel die 'vollständige hypergeometrische Funktion von der Ordnung p' ${}_{p+1}F_p$! Sie besitzt die Integraldarstellung§

$$ {}_{p+1}F_p\left[\begin{matrix} a_1,\ldots, a_{p+1}; \\ c_1,\ldots, c_p; \end{matrix} x\right] = \frac{\Gamma(c_1)\ldots\Gamma(c_p)}{\Gamma(a_1)\Gamma(c_1-a_1)\ldots\Gamma(a_p)\Gamma(c_p-a_p)} \times$$

$$\times \int_0^1 \ldots \int_0^1 t_1^{a_1-1}(1-t_1)^{c_1-a_1-1}\ldots t_p^{a_p-1}(1-t_p)^{c_p-a_p-1}(1-t_1\ldots t_p\, x)^{-a_{p+1}}dt_1\ldots dt_p$$

$$(0 < \Re(a_1) < \Re(c_1);\ldots;\quad 0 < \Re(a_p) < \Re(c_p)). \tag{1.5}$$

Da die verallgemeinerte hypergeometrische Reihe Permutationen der Parameter gegenüber invariant ist, so müssen alle $(p+1)!$ Integrale, die aus der rechten Seite von (1.5) durch Permutationen der Parameter $a_1,\ldots, a_{p+1}$ hervorgehen, übereinstimmen. Wie beweist man die Identität aller dieser Integrale unmittelbar?

Es können genau dieselben Wege eingeschlagen werden, die auch bei der Gaussschen Reihe zum Ziele führen. Die rechte Seite von (1.5) kann in ein $(2p+1)$-faches Integral übergeführt werden, welches in den a_i symmetrisch ist, und also in Evidenz setzt, dass die rechte Seite von (1.5) bei Permutationen der a_i sich nicht ändert. Aber auch der zweite Weg—die Konstruktion allgemeinerer Integralformeln—kann mit Erfolg beschritten werden.

† **2** Gleichung (4.1).

‡ Wie üblich bedeutet

$$(a)_0 = 1, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1) \quad (n = 1, 2,\ldots).$$

§ Vergl. z.B. **1** Seite 142 Gleichung (13).

In vorliegender Mitteilung werden die bei diesen Überlegungen verwendeten allgemeineren Funktionalbeziehungen für alle hypergeometrischen Reihen einer Veränderlichen—sowohl für die 'vollständigen' als auch für die 'konfluenten'—aufgestellt, und aus ihnen Integraldarstellungen für die hypergeometrischen Reihen abgeleitet. Die veröffentlichten Ergebnisse enthalten die Resultate der früheren Arbeit als Sonderfälle. Die aufgestellten Beziehungen hängen in leicht ersichtlicher Weise mit gewissen Formeln der Operatorenrechnung mit mehreren Veränderlichen zusammen, worauf jedoch der Kürze halber nicht ausführlich eingegangen werden soll.

2. Um die herzuleitenden Integralformeln kürzer anschreiben zu können, verwenden wir für die in der Operatorenrechnung häufig auftretende 'Faltung' zweier Funktionen die gebräuchliche Bezeichnung

$$f(x)*g(x) = \int_0^x f(u)g(x-u)\,du = x\int_0^1 f[tx]g[(1-t)x]\,dt. \tag{2.1}$$

Entsprechend falten wir Funktionen von mehreren Veränderlichen nach der Definitionsgleichung

$$\begin{aligned} &f(x_1,\dots,x_n)*\dots*g(x_1,\dots,x_n) \\ &= \int_0^{x_1}\dots\int_0^{x_n} f(u_1,\dots,u_n)g(x_1-u_1,\dots,x_n-u_n)\,du_1\dots du_n \\ &= x_1\dots x_n\int_0^1\dots\int_0^1 f[t_1x_1,\dots,t_nx_n]g[(1-t_1)x_1,\dots,(1-t_n)x_n]\,dt_1\dots dt_n. \end{aligned} \tag{2.2}$$

Eine der am häufigsten auftretenden Faltungsformeln ist das Eulersche Integral erster Gattung†

$$\frac{x^{a-1}}{\Gamma(a)}*\frac{x^{b-1}}{\Gamma(b)} = \frac{x^{a+b-1}}{\Gamma(a+b)} \quad (0<\Re(a);\ 0<\Re(b)), \tag{2.3}$$

welche ein transzendentes Additionstheorem der Potenz zum Ausdruck bringt.

In diesem Abschnitt werden wir Beziehungen zwischen solchen hypergeometrischen Reihen herleiten, die in allen Parametern der 'zweiten Gruppe' übereinstimmen, bei denen also $c_j = c'_j$ für alle $j = 1,\dots, q$ ist, und nur einige—möglicherweise alle—der a_j in beiden Reihen verschieden sind. Beginnen wir mit der Beziehung, die

† 5 § 12.41.

zwischen zwei Reihen mit gemeinsamen $c_1,\ldots,c_q$ besteht, wenn auch die a_i bis auf eines—etwa a_1—übereinstimmen. Diese Beziehung lautet:

$$\frac{x^{\alpha_1-1}}{\Gamma(\alpha_1)}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,x\right]=\frac{x^{\alpha_1-a_1-1}}{\Gamma(\alpha_1-a_1)}*\frac{x^{a_1-1}}{\Gamma(a_1)}\,{}_pF_q\left[\begin{matrix}\alpha_1,a_2,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,x\right]$$

$$(0<\Re(a_1)<\Re(\alpha_1). \qquad (2.4)$$

Ist $p-q=1$, so ist überdies $|x|<1$ vorauszusetzen, während für $p-q\leqslant 0$ die Veränderliche x gar keiner Beschränkung unterliegt. Der Beweis dieser Beziehung erfolgt sehr einfach durch Entwicklung der hypergeometrischen Reihen auf beiden Seiten von (2.4) nach Potenzen von x und Anwendung von (2.3) auf der rechten Seite. Die hierbei vorgenommene gliedweise Integration ist unter den angegebenen Bedingungen sicher gestattet, da die hypergeometrische Reihe im Integrationsintervall, welches ganz im Inneren ihres Konvergenzkreises liegt, gleichmässig konvergiert.

Zwischen zwei Funktionen ${}_pF_q$ in denen mehrere der Parameter der 'ersten Gruppe'—etwa die Parameter $a_1,\ldots,a_m$—verschieden sind, besteht eine Beziehung, welche die Form einer mehrdimensionalen Faltung zeigt. Das Argument der hypergeometrischen Reihen ist das Produkt der Veränderlichen, nach denen zu falten ist. Die allgemeinste Formel dieser Gestalt ist

$$\frac{x_1^{\alpha_1-1}}{\Gamma(\alpha_1)}\cdots\frac{x_m^{\alpha_m-1}}{\Gamma(\alpha_m)}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,x_1\ldots x_m\right]$$
$$=\frac{x_1^{\alpha_1-a_1-1}}{\Gamma(\alpha_1-a_1)}\cdots\frac{x_m^{\alpha_m-a_m-1}}{\Gamma(\alpha_m-a_m)}*\ldots*\frac{x_1^{a_1-1}}{\Gamma(a_1)}\cdots\frac{x_m^{a_m-1}}{\Gamma(a_m)}\times$$
$$\times\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_m,a_{m+1},\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,x_1\ldots x_m\right]$$

$$(0<\Re(a_1)<\Re(\alpha_1);\ldots;\,0<\Re(a_m)<\Re(\alpha_m);\; m\leqslant p). \qquad (2.5)$$

Ausgeschrieben lautet diese Formel mit $x_1\ldots x_m=x$

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,x\right]=\frac{\Gamma(\alpha_1)}{\Gamma(a_1)\Gamma(\alpha_1-a_1)}\cdots\frac{\Gamma(\alpha_m)}{\Gamma(a_m)\Gamma(\alpha_m-a_m)}\times$$
$$\times\int_0^1\cdots\int_0^1 t_1^{a_1-1}(1-t_1)^{\alpha_1-a_1-1}\ldots t_m^{a_m-1}(1-t_m)^{\alpha_m-a_m-1}\times$$
$$\times\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_m,a_{m+1},\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}\,t_1\ldots t_m x\right]dt_1\ldots dt_m. \qquad (2.6)$$

Diese Formel stellt den—vermutlich einfachsten—Zusammenhang zwischen zwei hypergeometrischen Reihen dar, deren 'c-Parameter' übereinstimmen.

3. Durch entsprechende Wahl der Parameter α_i kann erreicht werden, dass sich die Ordnung der hypergeometrischen Funktion auf der rechten Seite von (2.5) bzw. (2.6) reduziert. Setzen wir etwa $\alpha_i = c_i$ $(i = 1,..., m)$, so ergibt sich wegen

$$ {}_pF_q\left[\begin{matrix} c_1,..., c_m, a_{m+1},..., a_p; \\ c_1,..., c_q; \end{matrix} x\right] = {}_{p-m}F_{q-m}\left[\begin{matrix} a_{m+1},..., a_p; \\ c_{m+1},..., c_q; \end{matrix} x\right] $$

mit der Abkürzung

$$ \phi = \prod_{i=1}^{m} \frac{\Gamma(c_i)}{\Gamma(a_i)\Gamma(c_i-a_i)} t_i^{a_i-1}(1-t_i)^{c_i-a_i-1} $$

$$ (0 < \Re(a_i) < \Re(c_i);\ i = 1,..., m) \qquad (3.1) $$

aus (2.6) folgende Integraldarstellung der hypergeometrischen Funktion

$$ {}_pF_q\left[\begin{matrix} a_1,..., a_p; \\ c_1,..., c_q; \end{matrix} x\right] = \int_0^1 ... \int_0^1 \phi\, {}_{p-m}F_{q-m}\left[\begin{matrix} a_{m+1},..., a_p; \\ c_{m+1},..., c_q; \end{matrix} t_1...t_m x\right] dt_1...dt_m $$

$$ (m \leqslant p;\ m \leqslant q). \qquad (3.2) $$

Einige Sonderfälle dieser Integraldarstellung verdienen besondere Beachtung. Wir betrachten zunächst die vollständige hypergeometrische Funktion von der Ordnung q, für welche $p = q+1$ ist und setzen $m = q$. Im Integranden auf der rechten Seite von (3.2) tritt dann die hypergeometrische Reihe

$$ {}_1F_0(a_{m+1}; z) = (1-z)^{-a_{m+1}} \qquad (3.3) $$

auf und ergibt für die vollständige hypergeometrische Reihe die bekannte Integraldarstellung

$$ {}_{m+1}F_m\left[\begin{matrix} a_1,..., a_{m+1}; \\ c_1,..., c_m; \end{matrix} x\right] = \int_0^1 ... \int_0^1 (1-t_1...t_m x)^{-a_{m+1}} \phi\, dt_1...dt_m. \qquad (3.4) $$

Im Falle der konfluenten hypergeometrischen Funktion $p = q$ setzen wir $m = p = q$. Wegen

$$ {}_0F_0(z) = e^z \qquad (3.5) $$

ergibt sich die Integraldarstellung

$$ {}_mF_m\left[\begin{matrix} a_1,..., a_m; \\ c_1,..., c_m; \end{matrix} x\right] = \int_0^1 ... \int_0^1 e^{t_1...t_m x} \phi\, dt_1...dt_m. \qquad (3.6) $$

Bei der 'bikonfluenten Funktion' $p+1 = q$ setzen wir $m = p$. Wegen

$$ {}_0F_1(2\nu+1;z) = \Gamma(2\nu+1)z^{-\nu}I_{2\nu}(2\sqrt{z})\dagger \tag{3.7}$$

erhalten wir die Integraldarstellung

$$ {}_mF_{m+1}\begin{bmatrix} a_1,\dots,a_m; & x \\ c_1,\dots,c_{m+1}; & \end{bmatrix}$$

$$= \Gamma(c_{m+1}) \int\limits_0^1 \dots \int\limits_0^1 (t_1\dots t_m x)^{-\frac{1}{2}(c_{m+1}-1)} I_{c_{m+1}-1}\{2\sqrt{(t_1\dots t_m x)}\}\phi\, dt_1\dots dt_m. \tag{3.8}$$

Schliesslich wollen wir noch die entsprechende Integraldarstellung für die 'mehrfach konfluente Funktion' oder 'hypergeometrische Funktion von der Ordnung m und Klasse $k-1$', $p = m$, $q = k+m$ angeben:

$$ {}_mF_{m+k}\begin{bmatrix} a_1,\dots,a_m; & x \\ c_1\dots,c_{m+k}; & \end{bmatrix} = \int\limits_0^1 \dots \int\limits_0^1 {}_0F_k(c_{m+1},\dots,c_{m+k};\, t_1\dots t_m x)\phi\, dt_1\dots dt_m. \tag{3.9}$$

In allen Integraldarstellungen dürfen die Parameter der ersten Gruppe untereinander (oder auch die Parameter der zweiten Gruppe untereinander) einer beliebigen Permutation unterworfen werden.

4. Wir wollen kurz die gewöhnliche hypergeometrische Reihe ${}_2F_1$ betrachten und zeigen, dass die Umformung ihrer beiden Integraldarstellungen (1.2) und (1.3) ineinander anstatt des von Wirtinger‡ verwendeten *dreifachen* Integrals mit Hilfe eines *Doppelintegrals* erfolgen kann, welches sich überdies als Sonderfall von (2.6) entpuppt. Um dies zu zeigen, setzen wir in (1.2) unter der weiteren Voraussetzung $0 < \Re(a) < \Re(c)$ ein

$$(1-xt)^{-a} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int\limits_0^1 \tau^{a-1}(1-\tau)^{c-a-1}(1-xt\tau)^{-c}\, d\tau, \tag{4.1}$$

und erhalten das in a und b symmetrische Doppelintegral

$$ {}_2F_1(a,b;c;x) = \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \times$$

$$\times \int\limits_0^1 \int\limits_0^1 t^{b-1}\tau^{a-1}(1-t)^{c-b-1}(1-\tau)^{c-a-1}(1-xt\tau)^{-c}\, dt d\tau. \tag{4.2}$$

† $I_\nu(w)$ bedeutet die Zylinderfunktion erster Art von rein imaginärem Argument. Vergl. **4** Seite 77 f.

‡ Loc. cit.

(1.3) entsteht nun, wenn in (4.2) zuerst die Integration nach t durchgeführt wird. Das Doppelintegral (4.2) erhalten wir auch, wenn wir in (2.6)

$$p = m = 2, \quad q = 1, \quad a_1 = a, \quad a_2 = b, \quad \alpha_1 = \alpha_2 = c_1 = c$$

setzen und (3.3) heranziehen. Diese vereinfachte Umformung von (1.2) in (1.3) nach der Riemann-Wirtingerschen Methode dürfte aber auch unabhängig von seiner Verbindung mit der allgemeineren Funktionalbeziehung (2.6) ein gewisses Interesse beanspruchen.

5. Als Gegenstück zu (2.5) wollen wir diejenige Beziehung angeben, welche zwei ${}_pF_q$ mit gleichen Parametern 'der ersten Gruppe' verbindet. Es besteht die Faltungsformel zurecht

$$\frac{x_1^{c_1-1}}{\Gamma(c_1)}\cdots\frac{x_n^{c_n-1}}{\Gamma(c_n)}\,{}_pF_q\begin{bmatrix}a_1,\dots,a_p;\\c_1,\dots,c_q;\end{bmatrix}x_1\dots x_n\Bigg]$$
$$= \frac{x_1^{c_1-\gamma_1-1}}{\Gamma(c_1-\gamma_1)}\cdots\frac{x_n^{c_n-\gamma_n-1}}{\Gamma(c_n-\gamma_n)}*\dots*\frac{x_1^{\gamma_1-1}}{\Gamma(\gamma_1)}\cdots\frac{x_n^{\gamma_n-1}}{\Gamma(\gamma_n)}\times$$
$$\times\,{}_pF_q\begin{bmatrix}a_1,\dots,a_p;\\\gamma_1,\dots,\gamma_n,c_{n+1},\dots,c_q;\end{bmatrix}x_1\dots x_n\Bigg]$$
$$(0<\Re(\gamma_1)<\Re(c_1);\dots;0<\Re(\gamma_n)<\Re(c_n);\ n\leqslant q). \tag{5.1}$$

Der Beweis dieser Formel erfolgt genau so wie der von (2.5) durch Entwicklung der hypergeometrischen Reihe, Anwendung von (2.3) und gliedweise Integration. Ausgeschrieben lautet diese Formel

$${}_pF_q\begin{bmatrix}a_1,\dots,a_p;\\c_1,\dots,c_q;\end{bmatrix}x\Bigg] = \frac{\Gamma(c_1)}{\Gamma(\gamma_1)\Gamma(c_1-\gamma_1)}\cdots\frac{\Gamma(c_n)}{\Gamma(\gamma_n)\Gamma(c_n-\gamma_n)}\times$$
$$\times\int_0^1\dots\int_0^1 t_1^{\gamma_1-1}(1-t_1)^{c_1-\gamma_1-1}\dots t_n^{\gamma_n-1}(1-t_n)^{c_n-\gamma_n-1}\times$$
$$\times\,{}_pF_q\begin{bmatrix}a_1,\dots,a_p;\\\gamma_1,\dots,\gamma_n,c_{n+1},\dots,c_q;\end{bmatrix}t_1\dots t_n\,x\Bigg]dt_1\dots dt_n. \tag{5.2}$$

Durch geeignete Spezialisierung der γ_j, nämlich durch die Annahmen $\gamma_j = a_j$, lassen sich auch aus dieser Beziehung die in Abschnitt 3 behandelten Integraldarstellungen herleiten.

6. Durch Kombination der beiden Relationen (2.6) und (5.2) lässt sich zwischen je zwei hypergeometrischen Funktionen bei festem $p-q$ eine Beziehung herstellen. Alle derartigen Beziehungen sind

enthalten in der allgemeinen Formel

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ c_1,\ldots,c_q; \end{matrix} x\right] = \frac{\Gamma(\alpha_1)}{\Gamma(a_1)\Gamma(\alpha_1-a_1)}\cdots\frac{\Gamma(\alpha_m)}{\Gamma(a_m)\Gamma(\alpha_m-a_m)}\times $$

$$ \times\frac{\Gamma(c_1)}{\Gamma(\gamma_1)\Gamma(c_1-\gamma_1)}\cdots\frac{\Gamma(c_n)}{\Gamma(\gamma_n)\Gamma(c_n-\gamma_n)}\int\limits_0^1\cdots\int\limits_0^1 u_1^{a_1-1}(1-u_1)^{\alpha_1-a_1-1}\cdots $$

$$ \cdots u_m^{a_m-1}(1-u_m)^{\alpha_m-a_m-1}v_1^{\gamma_1-1}(1-v_1)^{c_1-\gamma_1-1}\cdots v_n^{\gamma_n-1}(1-v_n)^{c_n-\gamma_n-1}\times $$

$$ \times{}_pF_q\left[\begin{matrix} \alpha_1,\ldots,\alpha_m,a_{m+1},\ldots,a_p; \\ \gamma_1,\ldots,\gamma_n,c_{n+1},\ldots,c_q; \end{matrix} u_1\ldots u_m v_1\ldots v_n x\right]du_1\ldots du_m\,dv_1\ldots dv_n $$

$$ \begin{pmatrix} 0<\Re(a_1)<\Re(\alpha_1);\ldots;0<\Re(a_m)<\Re(\alpha_m);m\leqslant p \\ 0<\Re(\gamma_1)<\Re(c_1);\ldots;0<\Re(\gamma_n)<\Re(c_n);\ n\leqslant q \end{pmatrix}. \tag{6.1} $$

Diese soeben abgeleitete Formel gestattet auch dasjenige $(p+q)$-fache in den Parametern symmetrische Integral zu gewinnen, welches als Verallgemeinerung des Wirtingerschen dreifachen Integrals für die Gausssche hypergeometrische Reihe aufgefasst werden muss.

Nehmen wir an, dass für alle Zeigerpaare i, j

$$ 0<\Re(a_i)<\Re(c_j) $$

sei! Dann lässt sich eine den Ungleichungen

$$ \Re(a_i)<\lambda<\Re(c_j)\quad (i=1,\ldots,p; j=1,\ldots,q) $$

genügende positive Konstante λ bestimmen. Die in Rede stehende symmetrische Integraldarstellung erhalten wir, wenn wir in (6.1)

$$ m=p,\quad n=q,\quad \alpha_1=\ldots=\alpha_m=\gamma_1=\ldots=\gamma_n=\lambda $$

setzen. Wir betrachten wieder zuerst die vollständige hypergeometrische Funktion von der Ordnung n. Wegen (3.3) ergibt sich für sie

$$ {}_{n+1}F_n\left[\begin{matrix} a_1\ldots,a_{n+1}; \\ c_1,\ldots,c_n; \end{matrix} x\right] $$

$$ =\frac{\Gamma(c_1)\ldots\Gamma(c_n)\Gamma(\lambda)}{\Gamma(a_1)\Gamma(\lambda-a_1)\ldots\Gamma(a_{n+1})\Gamma(\lambda-a_{n+1})\Gamma(c_1-\lambda)\ldots\Gamma(c_n-\lambda)}\times $$

$$ \times\int\limits_0^1\cdots\int\limits_0^1 u_1^{a_1-1}(1-u_1)^{\lambda-a_1-1}\cdots u_{n+1}^{a_{n+1}-1}(1-u_{n+1})^{\lambda-a_{n+1}-1}v_1^{\lambda-1}(1-v_1)^{c_1-\lambda-1}\cdots $$

$$ \cdots v_n^{\lambda-1}(1-v_n)^{c_n-\lambda-1}(1-u_1\ldots u_{n+1}v_1\ldots v_n x)^{-\lambda}du_1\ldots du_{n+1}dv_1\ldots dv_n. \tag{6.2} $$

Genau so erhalten wir für die konfluente Reihe wegen (3.5)

$$ {}_nF_n\begin{bmatrix} a_1,\dots,a_n; \\ c_1,\dots,c_n; \end{bmatrix} x = \frac{\Gamma(c_1)\dots\Gamma(c_n)}{\Gamma(a_1)\Gamma(\lambda-a_1)\Gamma(c_1-\lambda)\dots\Gamma(a_n)\Gamma(\lambda-a_n)\Gamma(c_n-\lambda)}\times $$

$$ \times\int_0^1\dots\int_0^1 u_1^{a_1-1}(1-u_1)^{\lambda-a_1-1}v_1^{\lambda-1}(1-v_1)^{c_1-\lambda-1}\dots u_n^{a_n-1}(1-u_n)^{\lambda-a_n-1}\times $$

$$ \times v_n^{\lambda-1}(1-v_n)^{c_n-\lambda-1}e^{u_1v_1\dots u_nv_nx}\,du_1\dots du_n\,dv_1\dots dv_n. \qquad (6.3) $$

Ähnliche Integraldarstellungen ergeben sich für die bi- und mehrfach konfluenten Reihen.

7. Die in den Abschnitten 3 und 6 hergeleiteten Integraldarstellungen der verallgemeinerten hypergeometrischen Reihen geben nur für die vollständige und für die konfluente hypergeometrische Reihe zufriedenstellende Ergebnisse. In den Integraldarstellungen der mehrfach konfluenten Reihen treten noch die Funktionen ${}_0F_k$ auf, während von einer brauchbaren Integraldarstellung verlangt werden muss, dass sie im Integranden nur elementare Funktionen enthalte. Im Falle der bikonfluenten Reihe kann diese kleine Schwierigkeit allerdings leicht überwunden werden. Es genügt in (3.8) die Besselsche Funktion durch irgend eine ihrer zahlreichen Integraldarstellungen zu ersetzen, um für ${}_mF_{m+1}$ eine brauchbare Integraldarstellung zu erhalten. Um auch im Falle der mehrfach konfluenten Reihen ähnlich vorgehen zu können, werden wir daher nach einer entsprechend einfachen Integraldarstellung der Funktion ${}_0F_k$ Ausschau halten.

Die zur Verwendung gelangende Integraldarstellung werden wir aus gewissen allgemeineren Beziehungen zwischen hypergeometrischen Funktionen durch Besonderung herleiten. Die Beziehungen, die wir meinen, hängen wieder mit der Operatorenrechnung zusammen, nur sind es diesmal nicht Faltungsformeln, sondern Umkehrungen von Laplaceschen Transformierten. Als Ausgangspunkt dient die auf Hankel zurückgehende Integraldarstellung der Gammafunktion†

$$ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^u u^{-z}\,du, \qquad (7.1) $$

in welcher der Integrationsweg von $-\infty$ kommend den Punkt 0 in

† 5, § 12.

positivem Sinne umkreist und wieder nach $-\infty$ zurückkehrt. Entlang dieses Weges soll $|\arg u| < \pi$ sein.

Nun können wir sofort zeigen, dass

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}x\right] = \frac{\Gamma(\lambda)}{2\pi i}\int_{-\infty}^{(0+)} e^u\,{}_{p+1}F_q\left[\begin{matrix}a_1,\ldots,a_p,\lambda;\frac{x}{u}\\ c_1,\ldots,c_q;\end{matrix}\right]\frac{du}{u^\lambda} \quad (p \leqslant q) \qquad (7.2) $$

ist. Dabei muss der willkürliche Parameter λ von $0, -1, -2,\ldots$ verschieden sein. Ist $p = q$, so muss überdies der Integrationsweg etwa ausserhalb des Kreises $|u| = |x|$ verlaufen. Für $p < q$ ist eine derartige Beschränkung nicht notwendig. Unter diesen Voraussetzungen dürfen wir nämlich auf der rechten Seite die hypergeometrische Funktion nach Potenzen von x/u entwickeln und wegen der gleichmässigen Konvergenz der Reihe gliedweise integrieren. Auf Grund von (7.1) ergibt sich dann ohne weiteres die Richtigkeit von (7.2).

Genau auf dieselbe Weise zeigt man die Richtigkeit der Formel

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;\end{matrix}x\right] = \frac{\Gamma(\lambda_1)\ldots\Gamma(\lambda_n)}{(2\pi i)^n}\times $$

$$ \times\int_{-\infty}^{(0+)}\ldots\int_{-\infty}^{(0+)} e^{u_1+\ldots+u_n}\,{}_{p+n}F_q\left[\begin{matrix}a_1,\ldots,a_p,\lambda_1,\ldots,\lambda_n;\dfrac{x}{u_1\ldots u_n}\\ c_1,\ldots,c_q;\end{matrix}\right]\frac{du_1}{u_1^{\lambda_1}}\ldots\frac{du_n}{n_n^{\lambda_n}} $$

$$ (p+n \leqslant q+1; \lambda_j \neq 0, -1, -2,\ldots; j = 1,\ldots,n). \qquad (7.3) $$

Falls $p+n \leqslant q$, unterliegen die Integrationswege keiner Beschränkung. Ist aber $p+n = q+1$, so müssen sie so verlaufen, dass ständig $|u_1\ldots u_n| > |x|$ bleibt.

Setzen wir in (7.3) $p = 0$, $n = q$, $\lambda_j = c_j$, so erhalten wir wegen (3.5) die erwünschte Integraldarstellung von ${}_0F_n$ in der Gestalt

$$ {}_0F_n(c_1,\ldots,c_n;x) $$

$$ = \frac{\Gamma(c_1)\ldots\Gamma(c_n)}{(2\pi i)^n}\int_{-\infty}^{(0+)}\ldots\int_{-\infty}^{(0+)}\exp\left(u_1+\ldots+u_n+\frac{x}{u_1\ldots u_n}\right)\frac{du_1}{u_1^{c_1}}\ldots\frac{du_n}{u_n^{c_n}} $$

$$ (c_j \neq 0, -1, -2,\ldots; j = 1,\ldots,n). \qquad (7.4) $$

Für $n = 1$ ist diese Formel mit der Sonine-Sommerfeldschen Integraldarstellung der Besselschen Funktionen† identisch, und bildet im Falle $n > 1$ die Verallgemeinerung dieser—vielleicht wichtigsten—Integraldarstellung der Besselschen Funktionen.

† **4**, § 6.2, Gleichung (1).

Die Integraldarstellung für ${}_kF_{k+n}$ erhalten wir dadurch, dass wir in (3.9) ${}_0F_n$ durch die Integraldarstellung (7.4) ersetzen. Es ist

$$
{}_kF_{k+n}\left[\begin{matrix}a_1,\ldots,a_k;\\ c_1,\ldots,c_{k+n};\end{matrix}\,x\right]=\int\limits_0^1\ldots\int\limits_0^1\int\limits_{-\infty}^{(0+)}\ldots\int\limits_{-\infty}^{(0+)}\exp\left(u_1+\ldots+u_n+\frac{t_1\ldots t_k x}{u_1\ldots u_n}\right)\times
$$

$$
\times\prod_{i=1}^{k}\frac{\Gamma(c_i)}{\Gamma(a_i)\Gamma(c_i-a_i)}t_i^{a_i-1}(1-t_i)^{c_i-a_i-1}\,dt_i\prod_{j=1}^{n}\frac{\Gamma(c_{k+j})}{2\pi i}\frac{du_j}{u_j^{c_{k+j}}}
$$

$$
(0<\Re(a_i)<\Re(c_i);\ i=1,\ldots,k;\quad c_{k+j}\neq 0,-1,-2,\ldots;\ j=1,\ldots,n). \tag{7.5}
$$

Die in den Abschnitten 3, 5, 6 und 7 hergeleiteten Integraldarstellungen lassen ihre Invarianz gegenüber Permutationen der Parameter nicht mit voller Deutlichkeit erkennen. Diese Invarianz wäre dann klar zu Tage getreten, wenn wir z.B. bei der Ableitung von (3.2) nicht $\alpha_j = c_j$, sondern $\alpha_j = c'_j$ gesetzt hätten, unter $c'_1,\ldots,c'_q$ eine beliebige Permutation der $c_1,\ldots,c_q$ verstanden. Der im Text eingeschlagene Weg wurde der Einfachheit der Schreibweise wegen gewählt, zumal es auf Grund der soeben gemachten Bemerkung ganz klar ist, in welcher Weise die Invarianz der Integraldarstellungen gegenüber Permutationen der Parameter auch in der äusseren Gestalt der Formeln zum Ausdruck gebracht werden könnte.

Die über die Parametern gemachten Beschränkungen sind natürlich keinesfalls wesentlich. Sie wurden nur der Einfachheit halber aufrechterhalten und können jederzeit durch Einführung von Doppelschleifenintegralen entbehrlich gemacht werden.

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THE REPRESENTATION OF A NUMBER AS A SUM OF FOUR 'ALMOST EQUAL' SQUARES

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IN several papers* I have proved various results about the representation of numbers as the sum of s kth powers 'almost proportional' to assigned positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_3$. In a recent note, Auluck and Chowla† have improved my results for the particular case $k = 2$, $s = 4$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and given a very simple proof of the theorem:

If $n \not\equiv 0 \pmod 8$ *then*

$$n = x_1^2+x_2^2+x_3^2+x_4^2,$$

where the x *are integers satisfying*

$$|\tfrac{1}{4}n - x_i^2| = O(n^{\frac{3}{4}}) \quad (i = 1, 2, 3, 4).$$

I prove here that this is a 'best possible' result; that is, the $O(n^{\frac{3}{4}})$ in this theorem cannot be replaced by $o(n^{\frac{3}{4}})$. In fact, I prove the more general result:

If $k \geqslant 2$, $s \geqslant 1$ *and if* $\phi(n)$ *is any function of* n *such that* $\phi(n) \to 0$ *as* $n \to \infty$, *it is impossible for* $n = s(m^k + km^{k-1})$ *to be representable in the form*

$$n = x_1^k + x_2^k + \ldots + x_s^k \tag{1}$$

with

$$\left|\frac{n}{s} - x_i^k\right| < n^{1-\frac{1}{2k}}\phi(n) \quad (i = 1, 2, \ldots, s), \tag{2}$$

except for a finite number of values of m.

Without loss of generality we may suppose that $n^{\frac{1}{2k}}\phi(n) \to \infty$ as $n \to \infty$. We write

$$\phi(s(m^k + km^{k-1})) = \psi(m) = \psi,$$

so that $\psi \to 0$ and $m^{\frac{1}{2}}\psi \to \infty$ as $m \to \infty$. We use C to denote a positive number, not always the same at each occurrence, independent of m. We write

$$x_i = m + a_i \quad (i = 1, \ldots, s)$$

and suppose (1) and (2) true for $n = s(m^k + km^{k-1})$ and some m. Then (2) becomes

$$|m^k + km^{k-1} - x_i^k| < Cm^{k-\frac{1}{2}}\psi(m).$$

* *Phil. Trans. Roy. Soc.* A, 232 (1933), 1–26; *Math. Zeits.* 38 (1934), 730–46; *Quart. J. of Math.* (Oxford), 4 (1933), 37–51, 228–32, and 7 (1936), 230–40; *Proc. London Math. Soc.* (2), 42 (1937), 481–500.

† *Proc. Indian Acad. Sci.* 6 (1937), 81–2.

Hence $x_i \geqslant 0$* and

$$\begin{aligned} m^{k-1}|a_i| &= m^{k-1}|m-x_i| \\ &\leqslant |(m-x_i)(m^{k-1}+m^{k-2}x_i+\ldots+x_i^{k-1})| = |m^k-x_i^k| \\ &\leqslant |m^k+km^{k-1}-x_i^k|+km^{k-1} \\ &< Cm^{k-\frac{1}{2}}\psi(m). \end{aligned}$$

Hence $$|a_i| < Cm^{\frac{1}{2}}\psi(m). \tag{3}$$

Next, (1) becomes

$$n = s(m^k+km^{k-1}) = \sum_{i=1}^{s}(m+a_i)^k \tag{4}$$

and so

$$km^{k-1}\Big(s-\sum_{i=1}^{s}a_i\Big) = \sum_{i=1}^{s}\sum_{r=2}^{k}\binom{k}{r}m^{k-r}a_i^r,$$

$$\Big|s-\sum_{i=1}^{s}a_i\Big| < Cm^{-1}\sum_{i=1}^{s}a_i^2 < C\psi^2 < 1.$$

Hence $$\sum_{i=1}^{s}a_i = s. \tag{5}$$

Using this in (4), we obtain

$$\frac{k(k-1)}{1.2}m^{k-2}\sum_{i=1}^{s}a_i^2 = -\sum_{i=1}^{s}\sum_{r=3}^{k}\binom{k}{r}m^{k-r}a_i^r,$$

from which we have

$$m\sum_{i=1}^{s}a_i^2 \leqslant C\sum_{i=1}^{s}|a_i|^3.$$

Hence either, for at least one a_i,

$$|a_i| > Cm,$$

which contradicts (3), or

$$a_1 = a_2 = \ldots = a_s = 0,$$

which contradicts (5). Hence (2) is impossible for all sufficiently large m.

In the case of four squares ($k = 2$, $s = 4$) our result shows that, for $m > m_0$, the numbers

$$n = 4(m^2+2m)$$

are not representable in the form

$$n = x_1^2+x_2^2+x_3^2+x_4^2$$

with $$|\tfrac{1}{4}n-x_i^2| < n^{\frac{3}{4}}\phi(n). \tag{6}$$

If we take m odd, we have $n \equiv 4 \pmod 8$ and we see that $O(n^{\frac{3}{4}})$ in the theorem of Auluck and Chowla cannot be replaced by $o(n^{\frac{3}{4}})$.

* For sufficiently large m. This qualification is understood throughout the rest of the paper.

PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS (II)

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1. Equations with 'excessive' characteristics

IN an earlier paper under this title (**1**) I have given the solution, in the field of two independent variables, of those linear partial differential equations with constant coefficients which have just two distinct characteristics, extending the formulae to fields of more independent variables. I now consider those linear equations with constant coefficients in which the number of characteristics exceeds the number of independent variables (or, as I have proposed to say, the characteristics are 'excessive'). Certain new difficulties enter with this class of equation, and I shall begin, in the field of two independent variables, with the comparatively simple equation

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)V \equiv \prod_{r=1}^{m}\left(\frac{\partial}{\partial x}+b_r\frac{\partial}{\partial y}\right)V = V, \tag{1}$$

which has the m characteristics

$$y-b_r x = \text{constant} \quad (r = 1,..., m).$$

Now the problem presented by a partial differential equation can be envisaged in two forms apparently distinct: we can look for a 'general solution' embracing every possible solution, or we can seek to solve 'Cauchy's problem', in which we require the particular solution satisfying suitably-chosen 'data' (i.e. prescribed conditions) along an arbitrary 'base-curve'. Actually these two problems where solved (e.g. when the number of characteristics equals the number of independent variables) have turned out to be essentially the same. For the general solution is obtained by applying the solution of Cauchy's problem to a base-curve made up of a pair of characteristics through a convenient origin.

In these cases there may be some reason for preferring the general solution to the solution of Cauchy's problem as being the more 'natural' solution, since the arbitrary base-curve in Cauchy's problem introduces elements of arbitrariness additional to those arising from the equation itself. When, however, we come to equations with excessive characteristics the position seems to be reversed: the

characteristics suggest no 'natural' base-curve, and I shall therefore regard Cauchy's problem as the more fundamental.

2. The set of fundamental solutions

For the equation in two independent variables and with two characteristics Riemann's method consists essentially in applying Stokes's theorem to a differential expression, linear in a certain 'fundamental' solution u, taken round a contour formed by the pair of characteristics through an arbitrary point P and an arc of the base-curve. With m characteristics there are now $\frac{1}{2}m(m-1)$ such pairs and so $\frac{1}{2}m(m-1)$ possible contours. For each such contour there is the appropriate solution u_{rs} reducing to the single u when $m = 2$. Let us combine the results of the corresponding $\frac{1}{2}m(m-1)$ applications of Stokes's theorem by adding the various line-integrals: the surface-integrals characteristically disappear. We can take account of the *sense* of integration by attributing sign to the order of the symbols in u_{rs}, i.e. by defining

$$u_{sr} = -u_{rs} \tag{2}$$

and then integrating the differential expression linear in u_{rs} in the sense $PQ_r Q_s$, where PQ_r, PQ_s are the characteristics through P and $Q_r Q_s$ the arc of the base-curve.

A particular characteristic PQ_r through P will form part of $m-1$ contours, and the integral along PQ_r will involve $m-1$ of the functions u, namely $u_{r1},..., u_{r,r-1}, u_{r,r+1},..., u_{rm}$. In fact (as is easily seen) it will involve them only through their sum

$$U_r \equiv \sum_{s\neq r} u_{rs}.\text{*} \tag{3}$$

As regards the integrals along the base-curve it is convenient to take an arbitrary origin O on the curve and then to express them as a sum of integrals from O to Q_r ($r = 1,..., m$). In these integrals too the functions u_{rs} will then appear only through their partial sums U_r. Actually, by avoiding the appeal to Stokes's theorem, I work only with the m functions U_r, which I shall call the *set of fundamental solutions*. The $\frac{1}{2}m(m-1)$ subsidiary functions u_{rs} can, if necessary, be called the *sub-fundamental solutions*. It follows from (2), (3) that

$$\sum_{r=1}^{m} U_r = 0. \tag{4}$$

I shall refer to this as the *null condition*.

* I shall continually use the suffix '$s \neq r$' to denote sums and products taken over a set of values $s = 1,..., r-1, r+1,..., m$.

I take the set of fundamental solutions to be the functions*

$$U_r(X,Y;x,y) \equiv U_r(X-x,\ Y-y)$$
$$\equiv \frac{1}{2\pi i}\int\limits_{\Gamma_r}\sum_{n=0}^{\infty}\frac{\{(X-x)t-(Y-y)\}^{mn+m-2}\,dt}{(mn+m-2)!\,\{(t-b_1)...(t-b_m)\}^{n+1}}, \tag{5}$$

where Γ_r is a contour in the t-plane enclosing the point $t = b_r$ but not the points $t = b_1,..., b_{r-1}, b_{r+1},..., b_m$. It may be noted that the infinite series occurring in (5) is an 'extended' Bessel's series of the form

$$\sum_{n=0}^{\infty}\frac{u^{mn+m-2}}{(mn+m-2)!\,v^{n+1}},$$

which can be written in conventional symbols as

$$\frac{u^{m-2}v^{-1}}{(m-2)!}\,{}_0F_{m-1}\left[\begin{matrix}(u/m)^m v^{-1};\\ \frac{m-1}{m},\ \frac{m+1}{m},\ \frac{m+2}{m},...,\ \frac{2m-2}{m}\end{matrix}\right].$$

3. Properties of the fundamental solutions

3.1. We observe in the first place that these functions (5) satisfy the null condition (4). For, since r enters (5) only through the contour Γ_r, the sum $\sum U_r$ is the integrand of (5) taken round the aggregate of the contours $\Gamma_1,..., \Gamma_m$. These can be combined into a single contour enclosing every pole of the integrand and then deformed into the 'infinite circle', along which the integrand vanishes, since it is of the order $t^{-2}\,dt$.

3.2. Again, in variables X, Y, the functions $U_r(X,Y;x,y)$ satisfy the differential equation (1), i.e. the equation

$$f\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)V = V. \tag{6}$$

For

$$f\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)\frac{\{(X-x)t-(Y-y)\}^{mn+m-2}}{(mn+m-2)!\,\{(t-b_1)...(t-b_m)\}^{n+1}}$$
$$= \frac{\{(X-x)t-(Y-y)\}^{mn-2}}{(mn-2)!\,\{(t-b_1)...(t-b_m)\}^{n}},$$

so that the operator changes the nth term of the series into the $(n-1)$th; and it annihilates the first term. In other words it changes

* I have no doubt that the set of fundamental solutions is unique, but my present argument is not weakened if there are possible alternatives, and I am content at this stage to leave the question open.

the infinite series into itself and so gives equation (6). Similarly, in variables (x, y), the same functions $U_r(X, Y; x, y)$ satisfy the adjoint equation

$$f\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)V = V, \tag{7}$$

since they involve the variables only through the differences $X-x$, $Y-y$. Thus we have

LEMMA 1. *In variables X, Y, the fundamental solutions $U_r(X, Y; x, y)$ are solutions of the given equation, and, in variables x, y, they are solutions of the adjoint equation.*

3.3. I consider now the values of $U_r(X, Y; x, y)$ and certain of its derivatives along the associated characteristic PQ_r, i.e. along the line

$$y-Y = b_r(x-X). \tag{8}$$

In (5) the substitution (8) introduces a factor $(t-b_r)^{mn+m-2}$ into the numerator of the $(n+1)$th term of the series. Thus, unless

$$mn+m-2 \leqslant n,$$

the singularity at $t = b_r$ disappears, and the term vanishes on integration round Γ_r. More generally, in a kth derivative the factor introduced into the numerator is $(t-b_r)^{mn+m-2-k}$, and so only those terms survive the substitution in which

$$mn+m-2-k \leqslant n,$$

i.e.

$$(m-1)n+m-2 \leqslant k.$$

Since $m \geqslant 2$ by hypothesis,* no term survives when $k < m-2$; and, when $k = m-2$, the series reduces to its first term $n = 0$. Thus, if ϕ is of order $m-2$,

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)U_r = \frac{1}{2\pi i}\int\limits_{\Gamma_r} \frac{\phi(t, -1)\, dt}{(t-b_1)...(t-b_m)}\, dt$$

$$= \frac{\phi(b_r, -1)}{\prod\limits_{s \neq r} (b_r-b_s)}, \quad \text{a constant.}$$

* Actually, in proposing to consider only 'excessive' characteristics in this paper, I have supposed $m > 2$. But, not unnaturally, the analysis will be found to include as well the case of 'sufficient' characteristics, i.e. the case in which $m = 2$.

Thus U_r and its derivatives of order not exceeding $m-2$ are constant along PQ_r and so, differentiating along this line, we have

LEMMA 2. $$\frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}$$

and its derivatives of order not exceeding $m-2$ vanish along PQ_r.

These special values of U_r along PQ_r hold also at the particular point P and we have further

LEMMA 3. *At P every fundamental solution U_r and its derivatives of order not exceeding $m-3$ all vanish; and, if ϕ is of order $m-2$,*

$$\phi\left(-\frac{\partial}{\partial x},-\frac{\partial}{\partial y}\right)U_r=\frac{\phi(b_r,-1)}{\prod\limits_{s\neq r}(b_r-b_s)}.$$

4. The adjoint relations

4.1. I now proceed as in (**1**) and take $V(x,y)$ to be any solution of the given equation

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)V=V;$$

we have seen above that, in variables x,y, every U_r is a solution of the adjoint equation

$$f\left(-\frac{\partial}{\partial x},-\frac{\partial}{\partial y}\right)U_r=U_r.$$

Thus $$U_rf\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)V=Vf\left(-\frac{\partial}{\partial x},-\frac{\partial}{\partial y}\right)U_r \tag{9}$$

and, by the defining property of adjoint operators, this can be written as

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y},\quad \text{i.e.}\quad M=\frac{\partial K}{\partial x},\quad N=\frac{\partial K}{\partial y}$$

for some M,N,K. This relation can, of course, be put into many equivalent forms, for to M,N we can add any $\partial L/\partial x$, $\partial L/\partial y$, where L involves derivatives of U_r, V of order not exceeding $m-2$. To select the particular forms I require I proceed as follows.

4.2. In the first place I use α,β for the operators $\partial/\partial x$, $\partial/\partial y$ acting only on V and its derivatives, and $-\alpha'$, $-\beta'$ for the same operators acting only on the U_r and their derivatives,* so that on mixed operands in U_r, V the full operators are

$$\frac{\partial}{\partial x}=\alpha-\alpha',\qquad \frac{\partial}{\partial y}=\beta-\beta'. \tag{10}$$

* Notice that the change from α,β to α',β' is exactly that which changes an operator or an equation (with constant coefficients) into its adjoint.

Then, writing for brevity

$$g_r(u,v) \equiv \prod_{s=1}^{r-1}(u+b_s v)\prod_{s=r+1}^{m}(u+b_s v) \equiv \frac{f(u,v)}{u+b_r v}, \tag{11}$$

in the notation of (1), I assert that the two definitions*

$$\frac{\partial A_r}{\partial x} \equiv \left[\frac{f(\alpha',\beta)-f(\alpha',\beta')}{\beta-\beta'}+b_r\{g_r(\alpha,\beta)-g_r(\alpha',\beta)\}\right]U_r V, \tag{12}$$

$$-\frac{\partial A_r}{\partial y} \equiv \left[\frac{f(\alpha,\beta)-f(\alpha',\beta)}{\alpha-\alpha'}-b_r\frac{\beta-\beta'}{\alpha-\alpha'}\{g_r(\alpha,\beta)-g_r(\alpha',\beta)\}\right]U_r V \tag{13}$$

are consistent. In virtue of (10) consistency requires

$$(\beta-\beta')\frac{\partial A_r}{\partial x}-(\alpha-\alpha')\frac{\partial A_r}{\partial y} = 0,$$

i.e., on reduction,

$$[f(\alpha,\beta)-f(\alpha',\beta')]U_r V = 0.$$

But this, in changed notation, is just the adjoint relation (9). Thus the definitions (12), (13) are consistent and they define A_r exactly, save for an additive constant (to which I refer later).

4.3. Again, substituting for f in terms of g_r from (11) and reducing, we can rewrite (12), (13) as

$$\frac{\partial A_r}{\partial x} = \left[b_r g_r(\alpha,\beta)+(\alpha'+b_r\beta')\frac{g_r(\alpha',\beta)-g_r(\alpha',\beta')}{\beta-\beta'}\right]U_r V,$$

$$-\frac{\partial A_r}{\partial y} = \left[g_r(\alpha,\beta)+(\alpha'+b_r\beta')\frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\right]U_r V,$$

so that

$$\frac{\partial A_r}{\partial x}+b_r\frac{\partial A_r}{\partial y} = \left[\frac{g_r(\alpha',\beta)-g_r(\alpha',\beta')}{\beta-\beta'}-b_r\frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\right](\alpha'+b_r\beta')U_r V.$$

Since g_r is of order $m-1$, this expression on the right can be written as a sum of derivatives of

$$\frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}$$

of order not exceeding $m-2$, and therefore, by Lemma 2, it vanishes along the associated characteristic PQ_r. In other words, $\frac{\partial A_r}{\partial x}+b_r\frac{\partial A_r}{\partial y}$

* I need scarcely point out that the operators in (12), (13), though fractional in form, are in fact integral.

vanishes along this characteristic, i.e. *each $A_r(x,y)$ is constant along its associated characteristic PQ_r.*

5. The general solution

5.1. In forming the general solution of the differential equation (1) I take an arbitrary origin O, with coordinates (x_0, y_0) say, on the given base-curve and, for each r, perform the integration $\int dA_r$ from O to P along the path $OQ_r P$, where OQ_r is an arc of the base-curve and $Q_r P$ is the characteristic associated with A_r. Since, as we have just seen, A_r is constant along this characteristic, we get

$$[A_r]_O^P = \int\limits_O^{Q_r} dA_r = \int\limits_O^{Q_r} \left(\frac{\partial A_r}{\partial x}\,dx + \frac{\partial A_r}{\partial y}\,dy\right).$$

Summing in r we have

$$\Big[\sum_{r=1}^m A_r\Big]_O^P = \sum_{r=1}^m \int\limits_O^{Q_r} \left(\frac{\partial A_r}{\partial x}\,dx + \frac{\partial A_r}{\partial y}\,dy\right), \tag{14}$$

where the integrals on the right are taken along the base-curve, and $\partial A_r/\partial x$, $\partial A_r/\partial y$ are given by (12), (13). It remains to find an expression for $\sum A_r$, and, in particular, to determine its value at P.

5.2. Now, on summation in r, (12) gives

$$\begin{aligned}\frac{\partial}{\partial x}\sum_{r=1}^m A_r &= \frac{f(\alpha',\beta)-f(\alpha',\beta')}{\beta-\beta'}\left(\sum_{r=1}^m U_r\right)V + \sum_{r=1}^m b_r\{g_r(\alpha,\beta)-g_r(\alpha',\beta)\}U_r V \\ &= \frac{\partial}{\partial x}\sum_{r=1}^m b_r \frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\,U_r\,V,\end{aligned}$$

in virtue of the null condition (4) and of the definitions (10). We similarly get from (13)

$$\frac{\partial}{\partial y}\sum_{r=1}^m A_r = \frac{\partial}{\partial y}\sum_{r=1}^m b_r \frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\,U_r\,V.$$

Profiting by the additive constant still unprescribed in the definitions of the A_r we may sufficiently write

$$\sum_{r=1}^m A_r = \sum_{r=1}^m b_r \frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\,U_r\,V. \tag{15}$$

At P, since g_r is of order $m-1$, we need retain on the right of (15) only

operators of highest order in virtue of Lemma 3,

$$\text{i.e.}\quad \sum_{r=1}^{m} A_r(X,Y) = \sum_{r=1}^{m} b_r \frac{g_r(\alpha',0)}{\alpha'} U_r V$$
$$= V(X,Y)\sum_{r=1}^{m} b_r \alpha'^{m-2} U_r$$
$$= V(X,Y)\sum_{r=1}^{m} \frac{b_r^{m-1}}{\prod_{s\neq r}(b_r-b_s)}, \quad \text{by Lemma 3,}$$
$$= V(X,Y), \quad \text{by simple algebra.}$$

5.3. Hence finally from (14) we have

$$V(X,Y) = \sum_{r=1}^{m} A_r(x_0,y_0) + \sum_{r=1}^{m} \int_{O}^{Q_r} \left(\frac{\partial A_r}{\partial x}\,dx + \frac{\partial A_r}{\partial y}\,dy\right), \tag{16}$$

where $\sum A_r$, $\partial A_r/\partial x$, $\partial A_r/\partial y$ are as defined in (15), (12), (13). This expresses the value, at any arbitrary point (X,Y), of any solution V of the given equation (1) in terms of its value (and of the values of its derivatives of orders less than m) along the arbitrary base-curve: that is to say (16) is the desired solution of Cauchy's problem for (1).

6. The general equation in two variables

6.1. I now extend the foregoing results to the general equation (with excessive characteristics) in the field of two independent variables. We may consider the equation in the form

$$F\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)V \equiv \left[\prod_{r=1}^{m}\left(\frac{\partial}{\partial x}+b_r\frac{\partial}{\partial y}\right) - \sum_{r=1}^{m} f_{m-r}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\right]V = 0, \tag{17}$$

where f_s denotes a polynomial of order s (and, in particular, f_0 is a mere constant). For this extended equation the fundamental solutions can be written in the form

$$U_r(X,Y;x,y) \equiv \frac{1}{2\pi i}\int_{\Gamma_r} \sum \frac{\{(X-x)t-(Y-y)\}^{M+m-2}}{(M+m-2)!} \times$$
$$\times \prod_{s=1}^{m} \frac{\{f_{m-s}(t,-1)\}^{\nu_s}}{\nu_s!}\,\frac{N!\,dt}{\{(t-b_1)\ldots(t-b_m)\}^{N+1}}. \tag{18}$$

Here Γ_r is still a simple contour round the point $t = b_r$ only, the summation $\sum$ is taken over all positive integer and zero values of $\nu_1,\ldots,\nu_m$, and

$$M \equiv \sum_{s=1}^{m} s\nu_s, \qquad N \equiv \sum_{s=1}^{m} \nu_s. \tag{19}$$

In (18) the order in t of the integrand (including the differential element dt) is

$$M+m-1+\sum_{s=1}^{m}(m-s)\nu_s-m(N+1) = -1, \text{ by (19).}$$

Thus, as in § 3.1, the integrand vanishes round the infinite circle, and the extended fundamental solutions (18) still satisfy the null condition

$$\sum_{r=1}^{m} U_r = 0. \tag{20}$$

I go on to extend to these solutions the properties of Lemmas 1–3 established in § 3 for the simple fundamental solutions (5).

6.2. To show that in variables (X, Y) they are solutions of the differential equation (17) I use the argument of (1) § 4.*

If in the integrand of (18) we write $\nu_\sigma \equiv \nu'_{\sigma-1}$ for a particular $s = \sigma$ and drop accents, the integrand becomes

$$\sum \frac{\{(X-x)t-(Y-y)\}^{M+m-\sigma-2}}{(M+m-\sigma-2)!}\frac{\nu_\sigma}{Nf_{m-\sigma}(t,-1)}\prod_{s=1}^{m}\frac{\{f_{m-s}(t,-1)\}^{\nu_s}N!}{\nu_s!\{(t-b_1)...(t-b_m)\}^N}.$$

In the summation $\sum$ zero values of ν_σ are now excluded by definition, but in presence of the factor ν_σ we can add them in and so retain the original definition of $\sum$. Operation with $f_{m-\sigma}(\partial/\partial X, \partial/\partial Y)$ on this integrand then gives

$$\sum \frac{\nu_\sigma}{N}\frac{\{(X-x)t-(Y-y)\}^{M-2}}{(M-2)!}\prod_{s=1}^{m}\frac{\{f_{m-s}(t,-1)\}^{\nu_s}}{\nu_s!}\frac{N!}{\{(t-b_1)...(t-b_m)\}^N}$$

$$\equiv \sum \frac{\nu_\sigma}{N}\Theta \quad \text{say,}$$

where Θ is independent of σ. Thus

$$\sum_{s=1}^{m} f_{m-s}\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)U_r = \frac{1}{2\pi i}\int_{\Gamma_r}\sum\left(\sum_{s=1}^{m}\frac{\nu_s}{N}\right)\Theta\, dt$$

$$= \frac{1}{2\pi i}\int_{\Gamma_r}\sum \Theta\, dt, \quad \text{by (19).}$$

But this last expression is exactly

$$\prod_{s=1}^{m}\left(\frac{\partial}{\partial X}+b_s\frac{\partial}{\partial Y}\right)U_r,$$

and so $U_r(X, Y; x, y)$ satisfies the given equation (17) in variables X, Y. As in § 3.2, it satisfies the corresponding adjoint equation in

* There, at the foot of p. 282, the factor $\tau(\rho, 0)$ should clearly have been *inside* not *outside* the summation $\sum$.

variables x, y, since the adjoint equation comes from the given equation merely by changing the signs of the variables. Thus Lemma 1 holds for the extended equation (17) and its set of fundamental solutions (18).

6.3. The extended form of Lemma 2 is

LEMMA 4. $$U_r, \quad \frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}$$

and their derivatives of orders less than $m-2$ *vanish along the characteristic* PQ_r*; and, if* ϕ *is of order* $m-2$,

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)U_r = \frac{\phi(b_r, -1)E_r}{\prod\limits_{s\neq r}(b_r-b_s)},$$

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)\left(\frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}\right) = -\frac{\phi(b_r, -1)f_{m-1}(b_r, -1)E_r}{\left\{\prod\limits_{s\neq r}(b_r-b_s)\right\}^2},$$

where $$E_r \equiv \exp\frac{(X-x)f_{m-1}(b_r, -1)}{\prod\limits_{s\neq r}(b_r-b_s)}.$$

To prove this, observe that in (18) the substitution $Y-y = b_r(X-x)$ introduces $M+m-2$ factors $t-b_r$ into the numerator; that, in general, k differentiations reduce this number by k, but that differentiation along the characteristic PQ_r, i.e. the differentiation

$$\partial/\partial x+b_r\,\partial/\partial y,$$

leaves this number unaltered. Thus in the kth derivatives of U_r and $\partial U_r/\partial x+b_r\,\partial U_r/\partial y$ the substitution removes every factor $t-b_r$ from the denominator unless $M+m-k-2 \leqslant N$,

i.e. unless $$\sum_{s=2}^{m}(s-1)\nu_s+m-2 \leqslant k.$$

Thus along PQ_r all derivatives of U_r and $\partial U_r/\partial x+b_r\,\partial U_r/\partial y$ of order less than $m-2$ vanish; and, in the derivatives of order $m-2$, only those terms survive in $\sum$ in which $\nu_2,\ldots, \nu_m = 0$. This gives

$$M = N = \nu_1 \equiv \nu \text{ say},$$

and so, with ϕ of order $m-2$,

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)U_r = \frac{1}{2\pi i}\int\limits_{\Gamma_r}\sum_{\nu=0}^{\infty}\phi(t, -1)\frac{(X-x)^\nu\{f_{m-1}(t, -1)\}^\nu\,dt}{\nu!\,(t-b_r)\left\{\prod\limits_{s\neq r}(t-b_s)\right\}^{\nu+1}}$$

$$= \frac{\phi(b_r, -1)E_r}{\left\{\prod\limits_{s\neq r}(b_r-b_s)\right\}}$$

in the notation of the lemma. Differentiation along the characteristic at once gives

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)\left(\frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}\right) = -\frac{\phi(b_r-1)f_{m-1}(b_r, -1)E_r}{\left\{\prod_{s\neq r}(b_r-b_s)\right\}^2}.$$

This completes the proof of Lemma 4.

6.4. The extension of Lemma 3 follows at once as a corollary of Lemma 4, namely

Lemma 5. *U_r and all its derivatives of order less than $m-2$ vanish at P; and, if ϕ is of order $m-2$,*

$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)U_r = \frac{\phi(b_r, -1)}{\prod_{s\neq r}(b_r-b_s)}.$$

7. Solution of the general equation in two variables

7.1. In extension of § 4.2 I now define

$$\left.\begin{aligned}\frac{\partial A_r}{\partial x} &\equiv \left[\frac{F(\alpha',\beta)-F(\alpha',\beta')}{\beta-\beta'}+b_r\{g_r(\alpha,\beta)-g_r(\alpha',\beta)\}\right]U_r V\\ -\frac{\partial A_r}{\partial y} &\equiv \left[\frac{F(\alpha,\beta)-F(\alpha',\beta)}{\alpha-\alpha'}-b_r\frac{\beta-\beta'}{\alpha-\alpha'}\{g_r(\alpha,\beta)-g_r(\alpha',\beta)\}\right]U_r V\end{aligned}\right\}, \quad (21)$$

where F is defined in (17) and g_r in (11). It will be noted that these definitions (21) differ from (12), (13) only in the replacement of f by F. They are a consistent pair, since, as in § 4.2,

$$\frac{\partial}{\partial y}\left(\frac{\partial A_r}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial A_r}{\partial y}\right) = [F(\alpha,\beta)-F(\alpha',\beta')]U_r V = 0,$$

since V, U_r satisfy (17) and its adjoint respectively.

7.2. Again, as in § 4.3, on substituting for F we can rewrite these definitions as

$$\frac{\partial A_r}{\partial x} = \left[b_r g_r(\alpha,\beta)+(\alpha'+b_r\beta')\frac{g_r(\alpha',\beta)-g_r(\alpha',\beta')}{\beta-\beta'}-\right.$$

$$\left.-\sum_{s=1}^{m}\frac{f_{m-s}(\alpha,\beta)-f_{m-s}(\alpha,\beta')}{\beta-\beta'}\right]U_r V,$$

$$-\frac{\partial A_r}{\partial y} = \left[g_r(\alpha,\beta)+(\alpha'+b_r\beta')\frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}-\right.$$

$$\left.-\sum_{s=1}^{m}\frac{f_{m-s}(\alpha,\beta')-f_{m-s}(\alpha',\beta')}{\alpha-\alpha'}\right]U_r V,$$

so that

$$\frac{\partial A_r}{\partial x}+b_r\frac{\partial A_r}{\partial y}$$

$$=\left[\frac{g_r(\alpha',\beta)-g_r(\alpha',\beta')}{\beta-\beta'}-b_r\frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}\right](\alpha'+b_r\beta')U_r V-$$

$$-\sum_{s=1}^{m}\left[\frac{f_{m-s}(\alpha,\beta)-f_{m-s}(\alpha,\beta')}{\beta-\beta'}-b_r\frac{f_{m-s}(\alpha,\beta')-f_{m-s}(\alpha',\beta')}{\alpha-\alpha'}\right]U_r V. \quad (22)$$

Now g_r is of order $m-1$ and so, by Lemma 4, along the characteristic PQ_r we can reduce the first term in (22) to

$$-V\left[\frac{g_r(\alpha',0)-g_r(\alpha',\beta')}{-\beta'}-b_r\frac{g_r(\alpha',0)}{\alpha'}\right]\left(\frac{\partial U_r}{\partial x}+b_r\frac{\partial U_r}{\partial y}\right),$$

retaining only operators of order $m-2$ in α',β'. Again, by the same lemma, this can be written

$$V\left[\frac{g_r(b_r,0)-g_r(b_r,-1)}{1}-b_r\frac{g_r(b_r,0)}{b_r}\right]\frac{f_{m-1}(b_r,-1)E_r}{\left[\prod_{s\neq r}(b_r-b_s)\right]^2}$$

$$=-V\frac{f_{m-1}(b_r,-1)E_r}{\prod_{s\neq r}(b_r-b_s)},\quad \text{since}\quad g_r(b_r,-1)=\prod_{s\neq r}(b_r-b_s).$$

Again, in the summation forming the second term of (22) the same lemma shows that along PQ_r only the term $s=1$ survives and that this reduces to

$$V\left[-f_{m-1}(0,-1)+b_r\frac{f_{m-1}(0,-1)-f_{m-1}(b_r,-1)}{b_r}\right]\frac{E_r}{\prod_{s\neq r}(b_r-b_s)}$$

$$=-\frac{Vf_{m-1}(b_r,-1)E_r}{\prod_{s\neq r}(b_r-b_s)}.$$

Thus $\dfrac{\partial A_r}{\partial x}+b_r\dfrac{\partial A_r}{\partial y}$ vanishes along the characteristic PQ_r, and, as in § 5.1, we deduce the relation

$$\left[\sum_{r=1}^{m}A_r\right]_O^P=\sum_{r=1}^{m}\int_O^{Q_r}\left(\frac{\partial A_r}{\partial x}\,dx+\frac{\partial A_r}{\partial y}\,dy\right). \quad (23)$$

7.3. The argument of § 5.2 applies without modification to the extended functions A_r defined by (21), and, as there, we can write

$$\sum_{r=1}^{m}A_r=\sum_{r=1}^{m}b_r\frac{g_r(\alpha,\beta)-g_r(\alpha',\beta)}{\alpha-\alpha'}U_r V. \quad (24)$$

Continuing the argument as in § 5.2, we have once again at P

$$\sum_{r=1}^{m} A_r(X,Y) = V(X,Y).$$

7.4. Hence, finally, we obtain the solution of Cauchy's problem for the extended equation (17) in the form

$$V(X,Y) = \sum_{r=1}^{m} A_r(x_0,y_0) + \sum_{r=1}^{m} \int_{O}^{Q_r} \left(\frac{\partial A_r}{\partial x}\,dx + \frac{\partial A_r}{\partial y}\,dy\right), \tag{25}$$

where $\sum A_r$, $\partial A_r/\partial x$, $\partial A_r/\partial y$ are defined in (24)—or (15)—and (21). This form is symbolically the same as the earlier (16), the only difference being in the extended definition of $\partial A_r/\partial x$, $\partial A_r/\partial y$.

7.5. I make two remarks on these solutions (16), (25). In the first place, they extend to complex values of the constants and, in particular, of the b_r, provided only that a reasonable interpretation can then be given to the curvilinear integrals involved.

In the second place, the solution (16) remains valid even when the b_r are not all distinct. This, however, is evidently not true without qualification of the solution (25) owing to the denominator

$$\prod_{s \neq r} (b_s - b_r)$$

occurring in E_r and elsewhere in the enunciations of Lemmas 4, 5. Various sufficient conditions will suggest themselves, e.g. that the terms f_{m-1} be absent from the equation. Actually the solution (25) remains valid, I believe, so long as the associated algebraic curve $F(x,y) = 0$ has linear asymptotes; if the repeated factors give rise to parabolic asymptotes, the argument must be reconstructed. For the present, then, I exclude these 'parabolic' cases.

8. The simple equation in three variables

8.1. I pass on now to the field of three independent variables, which should sufficiently indicate how the foregoing analysis extends to the field of many independent variables. The analysis (up to the present) succeeds only with the special type of equations in which the operators of highest order factorize into linear factors; and, because of the awkwardness of excluding 'parabolic' forms, I consider here only the extension to this field of the simple equation (1). I take it in the form

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)V \equiv \prod_{r=1}^{m} \left(a_r\frac{\partial}{\partial x} + b_r\frac{\partial}{\partial y} + c_r\frac{\partial}{\partial z}\right)V = V, \tag{26}$$

where $m \geqslant 3$ and the factors of f are all distinct. In this extended field we have through an arbitrary point P, with coordinates (X, Y, Z), the m *characteristic edges* PQ_r

$$\frac{x-X}{a_r} = \frac{y-Y}{b_r} = \frac{z-Z}{c_r} \quad (r = 1,..., m) \tag{27}$$

along which are satisfied the m differential equations

$$\frac{dx}{a_r} = \frac{dy}{b_r} = \frac{dz}{c_r}. \tag{28}$$

These characteristic edges taken in pairs define the $\frac{1}{2}m(m-1)$ *characteristic planes* $PQ_r Q_s$ through P having equations

$$\begin{vmatrix} x-X & y-Y & z-Z \\ a_r & b_r & c_r \\ a_s & b_s & c_s \end{vmatrix} = 0 \quad (r, s = 1,..., m). \tag{29}$$

On the $m-1$ characteristic planes through a given edge PQ_r the partial differential equation

$$a_r \frac{\partial u}{\partial x} + b_r \frac{\partial u}{\partial y} + c_r \frac{\partial u}{\partial z} = 0 \tag{30}$$

is satisfied. For simplicity I consider only those equations in which all the characteristic edges and all the characteristic planes are distinct, so that no determinant $|a_r b_s c_t|$ vanishes.

8.2. There are now $\frac{1}{2}m(m-1)$ fundamental solutions corresponding to the $\frac{1}{2}m(m-1)$ characteristic planes. I write them in the form

$$U_{rs}(X, Y, Z; x, y, z)$$
$$= \frac{1}{(2\pi i)^2} \iint\limits_{(r,s)} \sum_{n=0}^{\infty} \frac{\{(X-x)t+(Y-y)u+(Z-z)\}^{mn+m-3}\, dtdu}{(mn+m-3)! \left\{\prod_{p=1}^{m} (a_p t+b_p u+c_p)\right\}^{n+1}}, \tag{31}$$

the integral being a double contour-integral* in which the integration in u (supposed taken first†) is round a contour which, for given t, includes $\tau_r \equiv a_r t+b_r u+c_r = 0$ and excludes $\tau_s \equiv a_s t+b_s u+c_s = 0$, and the subsequent integration in t is round a contour including the value of t given by $\tau_r = 0 = \tau_s$. This double contour of integration I have indicated by the suffix (r, s). If, alternatively, the u-contour

* See, for instance, (**1**) Chapter VI; (**2**) Chapter III; (**3**) 97–102.

† The order of integration is immaterial. (**1**) 165 foot, or, more generally, (**3**) 98 foot (§ 41).

includes $\tau_s = 0$ and excludes $\tau_r = 0$ the sign of the double integral is changed* so that

$$U_{sr} = -U_{rs}. \tag{32}$$

We can, of course, reduce the double integral (31) to a single contour integral by taking its residue (in u) at the pole $\tau_r = 0$: this gives

$$U_{rs} = \frac{1}{2\pi i}\int\limits_{\Gamma_{r,s}} dt \sum_{n=0}^{\infty}\left[\left(\frac{\partial}{\partial u}\right)^n \frac{\{(X-x)t+(Y-y)u+(Z-z)\}^{mn+m-3}}{(mn+m-3)!\,a_r^{n+1}\prod\limits_{p\neq r}(a_p t+b_p u+c_p)^{n+1}}\right]_{\tau_r=0} \tag{33}$$

where $\Gamma_{r,s}$ is a contour enclosing the point

$$t = t_s \equiv (b_r c_s - b_s c_r)/(c_r a_s - c_s a_r). \tag{34}$$

If now we fix r and sum in s from 1 to m omitting $s = r$, the integrand is invariable through the summation, and the aggregate of the contours $\Gamma_{r,s}$ can be combined to include the $m-1$ points t_s $(s \neq r)$, which are the only poles of the integrand, and then deformed to become the infinite circle. Since the degree in t of the integrand (including the differential element dt) is -1, the integrand vanishes along the infinite circle, and so

$$\sum_{s\neq r} U_{rs} = 0 \quad (r = 1,\ldots, m), \tag{35}$$

which is the form now taken by the null condition.

8.3. We see without difficulty that every U_{rs} satisfies the given equation

$$f\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right)V = V,$$

in variables X, Y, Z, and the adjoint equation

$$f\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)V = V,$$

in variables x, y, z.

8.4. Again, on the characteristic plane $PQ_r Q_s$ we can write, from (29),

$$\left.\begin{aligned} x-X &= a_r\lambda_r + a_s\lambda_s \\ y-Y &= b_r\lambda_r + b_s\lambda_s \\ z-Z &= c_r\lambda_r + c_s\lambda_s \end{aligned}\right\} \tag{36}$$

introducing parameters λ_r, λ_s, so that in a typical term of the summation in (31) the numerator becomes

$$(\lambda_r\tau_r + \lambda_s\tau_s)^{mn+m-3}, \tag{37}$$

* See (1) 165–6 for a particular example, though the principle is seen to apply generally.

where $$\tau_r \equiv a_r t + b_r u + c_r, \text{ etc.,} \tag{38}$$
as already defined. If we expand (37) as a sum of powers $(\tau_r)^p(\tau_s)^q$, where $p+q = mn+m-3$, every term in the double integral vanishes except those that retain *both* factors τ_r, τ_s in the denominator, i.e. those in which
$$p \leqslant n, \qquad q \leqslant n.$$
Adding, we have the *necessary* condition
$$mn+m-3 \leqslant 2n.$$
More generally, for a kth derivative of U_{rs} the corresponding condition is
$$(m-2)n+m-3 \leqslant k.$$
By hypothesis $m \geqslant 3$; hence every derivative of order less than $m-3$ vanishes, and, in those of order $m-3$, the summation reduces to its first term given by $n = 0$, which is then a mere constant. Thus U_{rs} and its derivatives of order not exceeding $m-3$ are constant on the characteristic plane $PQ_r Q_s$. Differentiating in this plane parallel to the characteristic edges PQ_r, PQ_s, i.e. operating with
$$\Delta_r' \equiv a_r \frac{\partial}{\partial x} + b_r \frac{\partial}{\partial y} + c_r \frac{\partial}{\partial z}, \text{ etc.,} \tag{39}$$
we see that $\Delta_r' U_{rs}$, $\Delta_s' U_{rs}$, $\Delta_r' \Delta_s' U_{rs}$ and their derivatives of order not exceeding $m-3$ vanish in the plane and, in particular, along the edges PQ_r, PQ_s. From these facts I select the lemmas I need, namely

LEMMA 6. *$\Delta_r' \Delta_s' U_{rs}$ and its derivatives of order not exceeding $m-3$ vanish on the characteristic plane $PQ_r Q_s$.*

LEMMA 7. *$\Delta_r' U_{rs}$, $\Delta_s' U_{rs}$ and their derivatives of order not exceeding $m-3$ vanish along the characteristic edges PQ_r, PQ_s.*

LEMMA 8. *U_{rs} and its derivatives of order less than $m-3$ vanish at P, and, if ϕ is of order $m-3$,*
$$\phi\left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) U_{rs} = \frac{1}{(2\pi i)^2} \iint\limits_{(r,s)} \frac{\phi(t,u,1)\, dt du}{\prod (at+bu+c)}.$$

9. The integrations

9.1. Now take $V(x,y,z)$ any solution of the given equation and, as in § 4.2, use α, β, γ for $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ acting only on V and its derivatives, and $-\alpha'$, $-\beta'$, $-\gamma'$ for the same operators acting only

on the U_{rs} and their derivatives, so that on mixed operands

$$\frac{\partial}{\partial x} = \alpha-\alpha', \qquad \frac{\partial}{\partial y} = \beta-\beta', \qquad \frac{\partial}{\partial z} = \gamma-\gamma'. \tag{40}$$

I now redefine (39) as

$$\Delta_r \equiv a_r \alpha+b_r \beta+c_r \gamma, \qquad \Delta_r' \equiv a_r \alpha'+b_r \beta'+c_r \gamma' \tag{41}$$

and write, as in (11),

$$g_r(u,v,w) \equiv \frac{f(u,v,w)}{a_r u+b_r v+c_r w}, \qquad g_{rs}(u,v,w) \equiv \frac{g_r(u,v,w)}{a_s u+b_s v+c_s w}. \tag{42}$$

I then define

$$\left.\begin{aligned}
\theta_{rs} &\equiv \frac{\partial B_{rs}}{\partial z}-\frac{\partial C_{rs}}{\partial y} \equiv \frac{f(\alpha,\beta,\gamma)-f(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V+\frac{\partial B'_{rs}}{\partial z}-\frac{\partial C'_{rs}}{\partial y} \\
\phi_{rs} &\equiv \frac{\partial C_{rs}}{\partial x}-\frac{\partial A_{rs}}{\partial z} \equiv \frac{f(\alpha',\beta,\gamma)-f(\alpha',\beta',\gamma)}{\beta-\beta'}U_{rs}V+\frac{\partial C'_{rs}}{\partial x}-\frac{\partial A'_{rs}}{\partial z} \\
\psi_{rs} &\equiv \frac{\partial A_{rs}}{\partial y}-\frac{\partial B_{rs}}{\partial x} \equiv \frac{f(\alpha',\beta',\gamma)-f(\alpha',\beta',\gamma')}{\gamma-\gamma'}U_{rs}V+\frac{\partial A'_{rs}}{\partial y}-\frac{\partial B'_{rs}}{\partial x}
\end{aligned}\right\}, \tag{43}$$

where

$$\left.\begin{aligned}
A'_{rs} &\equiv c_r\frac{g_r(\alpha',\beta,\gamma)-g_r(\alpha',\beta',\gamma)}{\beta-\beta'}U_{rs}V+ \\
&\qquad +c_s\frac{g_{rs}(\alpha',\beta,\gamma)-g_{rs}(\alpha',\beta',\gamma)}{\beta-\beta'}\Delta_r' U_{rs}V \\
B'_{rs} &\equiv -c_r\frac{g_r(\alpha,\beta,\gamma)-g_r(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V- \\
&\qquad -c_s\frac{g_{rs}(\alpha,\beta,\gamma)-g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'}\Delta_r' U_{rs}V \\
C'_{rs} &\equiv b_r\frac{g_r(\alpha,\beta,\gamma)-g_r(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V+ \\
&\qquad +b_s\frac{g_{rs}(\alpha,\beta,\gamma)-g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'}\Delta_r' U_{rs}V
\end{aligned}\right\}. \tag{44}$$

The definitions (43) have a single condition of consistency, namely

$$\frac{\partial\theta_{rs}}{\partial x}+\frac{\partial\phi_{rs}}{\partial y}+\frac{\partial\psi_{rs}}{\partial z} = 0,$$

and this is satisfied, since

$$\frac{\partial\theta_{rs}}{\partial x}+\frac{\partial\phi_{rs}}{\partial y}+\frac{\partial\psi_{rs}}{\partial z} = \{f(\alpha,\beta,\gamma)-f(\alpha',\beta',\gamma')\}U_{rs}V,$$

which vanishes by our usual argument.

9.2. Again, if we substitute in (43) from (42), (44), we can rewrite after much reduction

$$\left.\begin{aligned}
\theta_{rs} &= \Big[a_r g_r(\alpha,\beta,\gamma)+a_s g_{rs}(\alpha,\beta,\gamma)\Delta'_r+ \\
&\qquad\qquad +\frac{g_{rs}(\alpha,\beta,\gamma)-g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'}\Delta'_r\Delta'_s\Big]U_{rs}V \\
\phi_{rs} &= \Big[b_r g_r(\alpha,\beta,\gamma)+b_s g_{rs}(\alpha,\beta,\gamma)\Delta'_r+ \\
&\qquad\qquad +\frac{g_{rs}(\alpha',\beta,\gamma)-g_{rs}(\alpha',\beta',\gamma)}{\beta-\beta'}\Delta'_r\Delta'_s\Big]U_{rs}V \\
\psi_{rs} &= \Big[c_r g_r(\alpha,\beta,\gamma)+c_s g_{rs}(\alpha,\beta,\gamma)\Delta'_r+ \\
&\qquad\qquad +\frac{g_{rs}(\alpha',\beta',\gamma)-g_{rs}(\alpha',\beta',\gamma')}{\gamma-\gamma'}\Delta'_r\Delta'_s\Big]U_{rs}V
\end{aligned}\right\}. \tag{45}$$

Thus
$$H_{rs} \equiv \begin{vmatrix} a_r & a_s & \theta_{rs} \\ b_r & b_s & \phi_{rs} \\ c_r & c_s & \psi_{rs} \end{vmatrix} \tag{46}$$

$$= \begin{vmatrix} a_r & a_s & \dfrac{g_{rs}(\alpha,\beta,\gamma)+g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'} \\ b_r & b_s & \dfrac{g_{rs}(\alpha',\beta,\gamma)+g_{rs}(\alpha',\beta',\gamma)}{\beta-\beta'} \\ c_r & c_s & \dfrac{g_{rs}(\alpha',\beta',\gamma)-g_{rs}(\alpha',\beta',\gamma')}{\gamma-\gamma'} \end{vmatrix} \Delta'_r\Delta'_s U_{rs}V.$$

Since g_{rs} is of order $m-2$, this determinant can be expanded as a sum of derivatives of $\Delta'_r\Delta'_s U_{rs}$ of order not exceeding $m-3$, and so, by Lemma 6, it vanishes on the characteristic plane $PQ_r Q_s$,

i.e. *H_{rs} vanishes on the plane $PQ_r Q_s$.* (47)

Consider the double integrand

$$\Omega_{rs} \equiv \theta_{rs}\,dydz + \phi_{rs}\,dzdx + \psi_{rs}\,dxdy \tag{48}$$

taken over the characteristic plane $PQ_r Q_s$. Changing to the parameters λ_r, λ_s of (36) we have

$$dydz = \begin{vmatrix} b_r & b_s \\ c_r & c_s \end{vmatrix} d\lambda_r d\lambda_s, \text{ etc.},$$

so that on this plane

$$\Omega_{rs} = H_{rs}\,d\lambda_r d\lambda_s,$$

which vanishes by (47). Hence

$$\iint \Omega_{rs}$$
$$\equiv \iint \left\{\left(\frac{\partial B_{rs}}{\partial z}-\frac{\partial C_{rs}}{\partial y}\right) dydz+\left(\frac{\partial C_{rs}}{\partial x}-\frac{\partial A_{rs}}{\partial z}\right) dzdx+\left(\frac{\partial A_{rs}}{\partial y}-\frac{\partial B_{rs}}{\partial x}\right) dxdy\right\}$$

vanishes on the characteristic plane $PQ_r Q_s$.

9.3. Now let S be the base-surface carrying the data of the Cauchy's problem and let the plane $PQ_r Q_s$ cut S in the arc $Q_r Q_s$, the edges PQ_r, PQ_s cutting S in the points Q_r, Q_s, and consider the surface formed by the triangular area $PQ_r Q_s$ in the characteristic plane together with the triangular area $OQ_r Q_s$ on the surface S, where O is any convenient origin on S. This area has the boundary (part straight, part curved) $PQ_r OQ_s$. Apply Stokes's theorem

$$\iint \Omega_{rs} = -\int (A_{rs}\, dx + B_{rs}\, dy + C_{rs}\, dz)$$

to this surface and its boundary. On the left the surface-integral vanishes on the characteristic plane, as we have seen; on the right I split the integral into four parts corresponding to the four parts of the boundary. This gives

$$\iint\limits_{OQ_r Q_s} \Omega_{rs} = -\left(\int\limits_O^{Q_r} + \int\limits_{Q_r}^{P} - \int\limits_O^{Q_s} - \int\limits_{Q_s}^{P}\right)(A_{rs}\, dx + B_{rs}\, dy + C_{rs}\, dz),$$

where the sign of the surface-integral must be properly correlated with that of the line-integral. Summing for all distinct pairs r, s and interchanging the parameters r, s in the last two line-integrals, we have

$$\sum_{r,s} \iint\limits_{OQ_r Q_s} \Omega_{rs} = -\tfrac{1}{2}\sum_{r=1}^{m}\left(\int\limits_O^{Q_r} + \int\limits_{Q_r}^{P}\right)\sum_{s\neq r}\{(A_{rs}-A_{sr})\, dx +$$
$$+(B_{rs}-B_{sr})\, dy + (C_{rs}-C_{sr})\, dz\}. \quad (49)$$

9.4. As in § 5.2 the null condition (35) gives

$$\sum_{s\neq r}\left(\frac{\partial B_{rs}}{\partial z}-\frac{\partial C_{rs}}{\partial y}\right) = \sum_{s\neq r}\left(\frac{\partial B'_{rs}}{\partial z}-\frac{\partial C'_{rs}}{\partial y}\right), \quad \text{etc.,}$$

in the equations (43), so that we can sufficiently take

$$\sum_{s\neq r} A_{rs} = \sum_{s\neq r} A'_{rs}, \quad \text{etc.,}$$

and (49) then becomes

$$\sum_{r,s} \iint_{OQ_rQ_s} \Omega_{rs} = -\tfrac{1}{2} \sum_{r=1}^{m} \Big(\int_{O}^{Q_r} + \int_{Q_r}^{P} \Big) \sum_{s \neq r} \{(A'_{rs} - A'_{sr})\, dx +$$
$$+ (B'_{rs} - B'_{sr})\, dy + (C'_{rs} - C'_{sr})\, dz\}. \quad (50)$$

The integrals involving P are further reducible, for along PQ_r

$$\sum_{s \neq r} A'_{rs} = 0, \qquad \sum_{s \neq r} B'_{rs} = 0, \qquad \sum_{s \neq r} C'_{rs} = 0,$$

since on the right of (44) the second terms (involving $\Delta'_r U_{rs} V$) vanish by Lemma 7, and the first terms vanish, on summation, by the null condition (35).

Again along PQ_r

$$\frac{dx}{a_r} = \frac{dy}{b_r} = \frac{dz}{c_r} = d\lambda_r \quad \text{say,} \tag{51}$$

so that substituting again from (44) we have

$$\int_{Q_r}^{P} (A'_{sr}\, dx + B'_{sr}\, dy + C'_{sr}\, dz)$$
$$= \int_{Q_r}^{P} d\lambda_r \Big[-a_r c_s \frac{g_s(\alpha', \beta, \gamma) - g_s(\alpha', \beta', \gamma)}{\beta - \beta'} +$$
$$+ (b_r c_s - b_s c_r) \frac{g_s(\alpha, \beta, \gamma) - g_s(\alpha', \beta, \gamma)}{\alpha - \alpha'} \Big] U_{rs} V,$$

since $U_{sr} = -U_{rs}$ by (32). Using the definitions (41), (42) I find by algebraic manipulation that

$$\Big[-a_r c_s \frac{g_s(\alpha', \beta, \gamma) - g_s(\alpha', \beta', \gamma)}{\beta - \beta'} +$$
$$+ (b_r c_s - b_s c_r) \frac{g_s(\alpha, \beta, \gamma) - g_s(\alpha', \beta, \gamma)}{\alpha - \alpha'} \Big] U_{rs} V$$

can be rewritten

$$\Big[-a_r c_s \frac{g_r(\alpha', \beta, \gamma) - g_r(\alpha', \beta', \gamma)}{\beta - \beta'} +$$
$$+ a_r \frac{g_{rs}(\alpha', \beta, \gamma) - g_{rs}(\alpha', \beta', \gamma)}{\beta - \beta'} (c_r \Delta'_s - c_s \Delta'_r) +$$
$$+ (b_r c_s - b_s c_r) \frac{g_{rs}(\alpha, \beta, \gamma) - g_{rs}(\alpha', \beta, \gamma)}{\alpha - \alpha'} \Delta'_r +$$
$$+ (b_r c_s - b_s c_r) \frac{g_{rs}(\alpha, \beta, \gamma) - g_{rs}(\alpha', \beta, \gamma)}{\alpha - \alpha'} (\Delta_r - \Delta'_r) \Big] U_{rs} V.$$

By Lemma 7 the second and third terms vanish along PQ_r; by the null condition the first term vanishes on summation over $s \neq r$; and so we get

$$\sum_{s\neq r} \int\limits_{Q_r}^{P} (A'_{sr}\,dx + B'_{sr}\,dy + C'_{sr}\,dz)$$

$$= \sum_{s\neq r} (b_r c_s - b_s c_r) \int\limits_{Q_r}^{P} d\lambda_r \left\{(\Delta_r - \Delta'_r) \frac{g_{rs}(\alpha,\beta,\gamma) - g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'} U_{rs} V\right\}$$

$$= \sum_{s\neq r} (b_r c_s - b_s c_r) \int\limits_{Q_r}^{P} d\left[\frac{g_{rs}(\alpha,\beta,\gamma) - g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'} U_{rs} V\right],$$

since along PQ_r

$$(d\lambda_r)(\Delta_r - \Delta'_r) = (d\lambda_r)\{a_r(\alpha-\alpha') + b_r(\beta-\beta') + c_r(\gamma-\gamma')\}$$

$$= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}$$

by (40), (51). Hence (50) becomes

$$\sum_{r,s} \iint\limits_{OQ_rQ_s} \Omega_{rs} + \tfrac{1}{2}\sum_{r=1}^{m} \int\limits_{O}^{Q_r} \sum_{s\neq r} \{(A'_{rs} - A'_{sr})\,dx + (B'_{rs} - B'_{sr})\,dy + (C'_{rs} - C'_{sr})\,dz\}$$

$$= -\tfrac{1}{2}\sum_{r=1}^{m} \left\{\sum_{s\neq r} (b_r c_s - b_s c_r)\left[\frac{g_{rs}(\alpha,\beta,\gamma) - g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'} U_{rs} V\right]_{Q_r}^{P}\right\}. \tag{52}$$

9.5. Collecting the terms at P we get

$$-\tfrac{1}{2}\sum_{r=1}^{m} \sum_{r\neq s} (b_r c_s - b_s c_r) \frac{g_{rs}(\alpha,\beta,\gamma) - g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'} U_{rs} V$$

$$= -\tfrac{1}{2}V \sum_{r=1}^{m} \sum_{s\neq r} \frac{b_r c_s - b_s c_r}{(2\pi i)^2} \iint\limits_{(r,s)} \frac{g_{rs}(t,0,0)\,dtdu}{t\prod(at+bu+c)}, \quad \text{by Lemma 8,}$$

$$= -\frac{\tfrac{1}{2}V}{(2\pi i)^2} \sum_{r=1}^{m} \sum_{s\neq r} \iint\limits_{(r,s)} \frac{(b_r c_s - b_s c_r)t^{m-3}\,dtdu}{a_r a_s \prod\{t + (b/a)u + (c/a)\}}. \tag{53}$$

Now, by definition of the double contour (r,s), the double integral can be evaluated by taking first the residue of the integrand in u at $a_r t + b_r u + c_r = 0$ and then the residue in t at

$$t = (b_r c_s - b_s c_r)/(a_r b_s - a_s b_r).$$

We are thus at liberty to use this substitution for t in the integral and to write (53) as

$$-\frac{\frac{1}{2}V}{(2\pi i)^2}\sum_{r=1}^{m}\sum_{s\neq r}\iint\limits_{(r,s)}\left(\frac{b_s}{a_s}-\frac{b_r}{a_r}\right)\frac{t^{m-2}\,dtdu}{\prod\{t+(b/a)u+(c/a)\}}.$$

But, as we saw in § 8.2, $\displaystyle\iint\limits_{(r,s)} = -\iint\limits_{(s,r)}$,

and so, separating b_s/a_s and b_r/a_r in the above double integral and rearranging the summation we get

$$\frac{V}{(2\pi i)^2}\sum_{r=1}^{m}\sum_{s\neq r}\iint\limits_{(r,s)}\frac{b_r t^{m-2}\,dtdu.}{a_r\prod\{t+(b/a)u+(c/a)\}}.$$

Taking the residue at $u = -(a_r/b_r)t-(c_r/b_r)$ we then get

$$\frac{V}{2\pi i}\sum_{r=1}^{m}\sum_{s\neq r}\int\limits_{\Gamma_{rs}}\frac{t^{m-2}\,dt}{\prod\limits_{p\neq r}\{t(1-b'_p/b'_r)+(b_r c_p-b_p c_r)/a_p b_r\}},\tag{54}$$

where, as defined in § 8.2, the contour Γ_{rs} encloses the pole $t = t_s$ corresponding to the factor $p = s$ in the denominator of the integrand, and

$$b'_p \equiv b_p/a_p.\tag{55}$$

As usual, the $m-1$ contours given by the summation in s can be combined and deformed into the infinite circle. The denominator is of order $m-1$ and so, on passage to the limit, (54) reduces to

$$V\sum_{r=1}^{m}\frac{1}{\prod\limits_{p\neq r}(1-b'_p/b'_r)} = V\sum_{r=1}^{m}\frac{b'^{m-1}_r}{b'_r-b'_p} = V,$$

by simple algebra.*

Carrying back these reductions to (52) we now have

$$V(X,Y) = \tfrac{1}{2}\sum_{r=1}^{m}\sum_{s\neq r}(b_r c_s-b_s c_r)\left[\frac{g_{rs}(\alpha,\beta,\gamma)-g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V\right]_{Q_r}+$$

$$+\tfrac{1}{2}\sum_{r=1}^{m}\int\limits_{O}^{Q_r}\sum_{s\neq r}\{(A'_{rs}-A'_{sr})\,dx+(B'_{rs}-B'_{sr})\,dy+(C'_{rs}-C'_{sr})\,dz\}+$$

$$+\sum_{r,s}\iint\limits_{Q_rQ_s}\Omega_{rs},\tag{56}$$

* Much of the immediately foregoing analysis is awkward, and susceptible I am sure, of ultimate improvement.

where A'_{rs}, B'_{rs}, C'_{rs} are defined in (44) and Ω_{rs} in (48). This is the solution of Cauchy's problem for the equation (26).

By substitution for Ω_{rs} from (48), (43), and a second application of Stokes's theorem, we can write (56) in the simpler form

$$V(X,Y) = \tfrac{1}{2}\sum_{r=1}^{m}\sum_{s\neq r}(b_r c_s - b_s c_r)\left[\frac{g_{rs}(\alpha,\beta,\gamma)-g_{rs}(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V\right]_{Q_r}+$$

$$+\sum_{r,s}\int_{Q_r}^{Q_s}(A'_{rs}\,dx+B'_{rs}\,dy+C'_{rs}\,dz)+$$

$$+\sum_{r,s}\iint_{OQ_rQ_s}\left\{\left[\frac{f(\alpha,\beta,\gamma)-f(\alpha',\beta,\gamma)}{\alpha-\alpha'}U_{rs}V\right]dydz+\right.$$

$$+\left[\frac{f(\alpha',\beta,\gamma)-f(\alpha',\beta',\gamma)}{\beta-\beta'}U_{rs}V\right]dzdx+$$

$$\left.+\left[\frac{f(\alpha',\beta',\gamma)-f(\alpha',\beta',\gamma')}{\gamma-\gamma'}U_{rs}V\right]dxdy\right\}. \quad (57)$$

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THE EXCEPTIONAL VALUES OF FUNCTIONS WITH A NON-ENUMERABLE SET OF ESSENTIAL SINGULARITIES

By MARY L. CARTWRIGHT (*Cambridge*)

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1. LET $f(z)$ be a function which is meromorphic in an open domain D except for a set E of essential singularities. The object of this paper is to prove the following theorem and to discuss its relation to other known theorems of a similar type.

THEOREM 1. *If the set E is of linear measure zero, then $f(z)$ takes all values except perhaps a set of plane measure zero near each point of E.*

It is easy to see that, if E is of positive linear measure, $f(z)$ may omit a set of positive plane measure, in fact $f(z)$ may omit a whole region of the complex plane. I shall discuss later how far the result can be improved when E is of linear measure zero. The proof depends on the theory of minimal slit regions, and the complete result obtained can only be stated in terms of slit regions.

2. Besicovitch* proved that, *if E is of linear measure zero, then $f(z)$ is unbounded in the neighbourhood of each point of E.* It follows from this that the values which are omitted by $f(z)$ cannot contain a continuum Γ. For, if $f(z)$ omits a continuum Γ near z_0, a point of E, we can map the remaining part of the complex plane on the unit circle by means of the transformation $\zeta = \zeta(w)$; and then the function $\psi(z) = \zeta\{f(z)\}$ is regular and bounded near z_0. Hence by Besicovitch's theorem $\psi(z)$ is regular at z_0; and so $f(z)$ cannot have an essential singularity at z_0. However, this says nothing about the *measure* of the excepted values; for there are discrete sets of points with positive plane measure.

3. We need certain properties of parallel slit regions.† A parallel slit region is a region of the complex plane whose boundary consists only of straight lines parallel to a given direction, which we take to be the real axis unless otherwise stated. In what follows a slit

* A. S. Besicovitch, *Proc. London Math. Soc.* (2) 32 (1931), 1–9.

† See Hurwitz-Courant, *Funktionentheorie* (Berlin, 1929), 445–80, also P. Koebe, *Göttinger Nachrichten* (1909), 324–61, (1918), 60–71; and H. Grötzsch, *Berichte Math.-Phys. Kl. Sächs. Ak. Wiss. Leipzig*, 84 (1932), 15–36.

region always means a parallel slit region. If the slits are parallel to $\arg z = \phi$ or $\arg z = \phi+\pi$, we shall say that it is a ϕ-slit region. The slits may be points, in which case they can of course be considered as slits in any direction. We suppose that the point at infinity is an interior point of all the regions considered; and we say that a (1,1) conformal transformation is normal if it has the form

$$w(z) = z+\frac{a^{(1)}}{z}+\frac{a^{(2)}}{z^2}+\dots$$

near the point at infinity.

The fundamental theorem on the mapping of slit regions can be stated as follows: *Any region B of finite or infinite connectivity, having the point at infinity as an interior point, can be mapped by a normal* (1,1) *conformal transformation on a ϕ-slit region in such a way that the boundary of the slit region has plane measure zero,*

Let

$$w_1(z) = z+\frac{a_1^{(1)}}{z}+\frac{a_1^{(2)}}{z^2}+\dots,$$

$$w_2(z) = z+\frac{a_2^{(1)}}{z}+\frac{a_2^{(2)}}{z^2}+\dots,$$

be the mapping functions for slits in the directions of the real and imaginary axis respectively. Then the mapping function for the ϕ-slit region is

$$w(z) = \frac{\mu w_1+i\nu w_2}{\mu+i\nu}, \quad \text{where} \quad \frac{\mu}{\nu} = \tan\phi.$$

Grötzsch* has shown that $a_1^{(1)} = a_2^{(1)}$ if and only if the whole set of normal (1, 1) conformal transformations reduces to $w \equiv z$. In this case the region is a ϕ-slit region for $0 \leqslant \phi < \pi$, so that all the slits are points. Since the slit regions obtained in the fundamental theorem above have a certain minimal property defined by Koebe,† we shall call these regions *absolute minimal slit regions*. We shall also use the term to cover regions for which the point at infinity is not an interior point, provided that they can be transformed into absolute minimal slit regions by transformations of the form $w = 1/(z-c)$.

By the fundamental theorem just quoted the boundary of an absolute minimal slit region must have plane measure zero. It follows from Besicovitch's theorem that, if the boundary of a slit

* H. Grötzsch, loc. cit., Satz 3.

† P. Koebe, *Göttinger Nachrichten* (1918), 60–71.

region has linear measure zero, it is an absolute minimal slit region. If the boundary has positive linear measure, even though it may consist only of points, the region need not be an absolute minimal slit region;* but there are some absolute minimal slit regions whose boundaries have positive linear measure.†

4. We can now state Theorem 1 in a more complete form.

THEOREM 1′. *If $f(z)$ is one-valued and meromorphic in an open domain D except for a set E of essential singularities of linear measure zero, then the set of values taken by $f(z)$ form an absolute minimal slit region.*

Theorem 1 now appears as an immediate corollary of this theorem; for the boundary of an absolute minimal slit region has plane measure zero. But, since there are sets of points of positive linear measure which are not the boundary of an absolute minimal slit region, Theorem 1 does not include Theorem 1′.

Proof. Let z_0 be a point of E; and suppose that the values taken by $f(z)$ in $|z-z_0| < \delta$ form a region S_1 which is not an absolute minimal slit region. We have seen in § 2 that S_1 is an open region whose boundary consists only of discrete points. We may obviously suppose that S_1 contains the point at infinity. For, if not, we can consider the function

$$F(z) = \frac{1}{f(z)-f(z_1)},$$

where z_1 is a point of $|z-z_0| < \delta$ at which $f(z)$ is regular. Since S_1 is not an absolute minimal slit region, there is a non-identical transformation

$$\zeta(w) = w+\frac{a_1}{w}+\frac{a_2}{w^2}+\dots,$$

which maps the region S_1 of the w-plane on to a slit region S_0 of the ζ-plane. Let b be any interior point of S_0 other than the point at infinity, and let $c = \zeta(b)$. Then the function $\zeta(w)-c$ has a simple zero at $w = b$, and the function

$$\Phi(w) = \frac{\zeta(w)-c}{w-b}$$

is regular and bounded at all points of S_1. For $\Phi(w)$ tends to a finite

* See P. Koebe, *Göttinger Nachrichten* (1918), 60–71.

† See R. de Possel, *J. de l'École Polytechnique* (2), 30 (1932), 1–98, and H. Grötzsch, *Berichte Math.-Phys. Kl. Sächs. Ak. Wiss. Leipzig*, 84 (1932), 15 June.

limit as $w \to b$ and as $w \to \infty$; and $\Phi(w)$ is obviously bounded at all other points of S_1. Since $\zeta(w) \not\equiv w$, $\Phi(w)$ is not a constant.

Now consider the function

$$\psi(z) = \Phi\{f(z)\};$$

since $f(z)$ only takes values in S_1, $\psi(z)$ is regular for $|z-z_0| < \delta$ except at points of E. Since $\Phi(w)$ is bounded in S_1, $\psi(z)$ is bounded in $|z-z_0| < \delta$. It follows from Besicovitch's theorem that $\psi(z)$ is regular for $|z-z_0| < \delta$. But, since $f(z)$ has an essential singularity at z_0, $|f(z)|$ is unbounded near z_0; and, since $\Phi(w) \to 1$ as $|w| \to \infty$, $\psi(z) \to 1$ as $z \to z_0$ through some sequence of values. But, since $\psi(z)$ is regular at z_0, $\psi(z) \to 1$ uniformly as $z \to z_0$, i.e. $|w| = |f| \to \infty$ uniformly; and so $f(z)$ has a pole, which is a contradiction. Hence S_1 must be an absolute minimal slit region.

5. Since there are sets of points of positive linear measure which are the boundary of absolute minimal slit regions, as well as sets of positive linear measure which are not, we cannot state the result obtainable by this method completely in terms of Euclidean measure; but it is quite possible that other methods may lead to better results. Seidel* has shown that $f(z)$ may omit any set of logarithmic measure zero near certain sets of essential singularities of linear measure zero; and R. Nevanlinna† has shown that *if E is of absolute harmonic measure zero, then $f(z)$ takes all values except perhaps a set of absolute harmonic measure zero.* Harmonic measure zero is a property which depends on conformal representation of a region having the set as boundary points on the interior of the unit circle. A set of absolute harmonic measure zero always becomes a set of linear measure zero on the unit circle. A set of harmonic measure zero is much smaller than any set of finite positive dimensional measure;‡ but it includes all sets of logarithmic measure zero. Hence the set E in Theorem 1′ is much more general than in Nevanlinna's theorem, but Theorem 1′ allows a larger set of values to be omitted. The complete result in both cases depends on the geometrical configuration of the set of values omitted. The gap between the class of set which Theorem 1′ allows to be omitted and the set which Seidel has shown to be omitted is very large indeed.

* W. Seidel, *Trans. American Math. Soc.* 36 (1934), 218.

† R. Nevanlinna, *Eindeutige analytische Funktionen* (Berlin, 1936), 130–6; a definition of harmonic measure zero is given on pp. 106–14.

‡ See R. Nevanlinna, loc. cit. 142–53.

[*Added 4 November 1937.*] Since the above was written, Gillis and Erdős* have shown that any set of finite logarithmic measure has transfinite diameter zero, and therefore capacity zero and absolute harmonic measure zero.† More recently still Ursell‡ has shown that, when

$$\lim_{t \to 0} h(t) \log \frac{1}{t} = 0,$$

there are sets of finite h-measure whose transfinite diameter is not zero. It follows that these sets are not of absolute harmonic measure zero.

* *J. of London Math. Soc.* 12 (1937), 185.

† See Nevanlinna, loc. cit. 119, 128.

‡ *J. of London. Math. Soc.* (unpublished).

ON PRIMITIVE ROOTS IN FINITE FIELDS

By H. DAVENPORT (*Cambridge*)

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Introduction

IT was proved by Vinogradov† that, if p is a large prime, then the least positive primitive root $g \pmod p$ satisfies $g = O(p^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$. Let $P(x)$ be an irreducible polynomial $\pmod p$ of degree k, then a polynomial $f(x) \pmod p$ is said to be a primitive root $(\operatorname{mod} P(x))$‡ if the least positive integer ν for which

$$(f(x))^\nu \equiv 1 \pmod{P(x)}$$

is $\nu = p^k-1$. The method of Vinogradov can easily be adapted to prove that, if p is large, there is a primitive root $(\operatorname{mod} P(x))$ whose degree does not exceed k', where k' is the least integer greater than $\frac{1}{2}k$.

The object of this paper is to investigate to what extent a more precise result can hold.§ I shall prove:

THEOREM 1. *If $p > p_0(k)$, there exists a linear polynomial $x-a$ which is a primitive root* $(\operatorname{mod} P(x))$.

THEOREM 2.|| *For any given $p > 2$, there exist irreducible polynomials $P(x)$ such that no linear polynomial $ax+b$ is a primitive root* $(\operatorname{mod} P(x))$.

It is frequently convenient to use, instead of the language of irreducible polynomials $\pmod p$, the language of finite fields of p^k elements. An irreducible polynomial $P(x)$ of degree k provides a means of generating the finite field of p^k elements (denoted by $[p^k]$), the elements of the field being represented by the rational functions of a root ϑ of $P(x)$, with coefficients integers taken mod p. In the language of finite fields, Theorems 1 and 2 take the form:

THEOREM 1. *If $p > p_0(k)$, and ϑ is any given generating element of $[p^k]$, then there exists an integer a such that $\vartheta-a$ is a primitive root of $[p^k]$.*

THEOREM 2. *For any given $p > 2$ there exist fields $[p^k]$ and generating elements ϑ such that no element of $[p^k]$ of the form $a\vartheta+b$ is a primitive root of $[p^k]$.*

† The proof is given in Landau, *Vorlesungen über Zahlentheorie*, 2, 178–80.

‡ This is an abbreviation for $\operatorname{mod}(P(x), p)$.

§ The problem was suggested to me by Dr. H. L. Schmid.

|| For results on the distribution of irreducible polynomials $P(x)$ for which a given polynomial is a primitive root $(\operatorname{mod} P(x))$, see Billharz, *Math. Annalen*, 114 (1937), 476–92.

Proof of Theorem 1

In this section small Latin letters other than p, k, e, i, j, r, d denote integers taken mod p, and Greek letters other than χ, τ, ϕ, μ denote elements of $[p^k]$. In each case variables of summation take all values not explicitly excluded. The constants implied by the symbol O depend only on k and (in Lemma 1) r.

We recall some properties of finite fields and Gaussian sums. For any α, the Spur of α, denoted by $\operatorname{Sp}\alpha$, is an integer mod p, and has the property $\operatorname{Sp}(\alpha+\beta) = \operatorname{Sp}\alpha+\operatorname{Sp}\beta$. For any given $a_0,\ldots, a_{k-1}$ there is exactly one α for which

$$\operatorname{Sp}\alpha\vartheta^j = a_j \quad (j = 0, 1,\ldots, k-1).$$

We shall denote by χ a typical character of $[p^k]$, and define $\tau(\chi)$ by

$$\tau(\chi) = \sum_{\xi} \chi(\xi)e(\operatorname{Sp}\xi), \quad (e(a) = e^{2\pi ia/p}).$$

For any $\eta \neq 0$ we have

$$\begin{aligned}\chi(\eta)\tau(\bar{\chi}) &= \sum_{\xi} \bar{\chi}(\xi)\chi(\eta)e(\operatorname{Sp}\xi)\\ &= \sum_{\zeta} \bar{\chi}(\zeta)e(\operatorname{Sp}\eta\zeta),\end{aligned}$$

whence

$$\chi(\eta) = \frac{1}{\tau(\bar{\chi})}\sum_{\xi} \bar{\chi}(\xi)e(\operatorname{Sp}\xi\eta), \tag{1}$$

and this is obviously also valid for $\eta = 0$, provided that χ is not the principal character. Also, if χ is not the principal character, we have

$$\begin{aligned}|\tau(\chi)|^2 &= \sum_{\xi}\sum_{\eta\neq 0} \chi(\xi)\bar{\chi}(\eta)e\{\operatorname{Sp}(\xi-\eta)\}\\ &= \sum_{\zeta}\sum_{\eta\neq 0} \chi(\zeta)e\{\operatorname{Sp}\eta(\zeta-1)\}\\ &= (p^k-1)\sum_{\zeta=1}\chi(\zeta)+(-1)\sum_{\zeta\neq 1}\chi(\zeta)\\ &= p^k,\end{aligned}$$

proving the familiar result

$$|\tau(\chi)| = p^{\frac{1}{2}k}. \tag{2}$$

Let r be any positive integer, and let

$$T(a_1,\ldots, a_r) = \sum_{x_1,\ldots,x_r} e(a_1 s_1+\ldots+a_r s_r),$$

where $s_1 = x_1+\ldots+x_r,\ldots,\ s_r = x_1\ldots x_r$ are the elementary symmetric functions of $x_1,\ldots, x_r$. Define $a_1,\ldots, a_r$ and $b_1,\ldots, b_r$ to be equivalent if there exists a $u \neq 0$ such that $b_j = a_j u^j$ for $j = 1, 2,\ldots, r$. This

relation has the usual properties of an equivalence, and the sum T has the same value for equivalent a's.

LEMMA 1. $\sum^*_{a_1,\ldots,a_r} |T(a_1,\ldots,a_r)|^2 = O(p^{2r-1})$, *where the * indicates that the summation is over one out of each set of equivalent a's other than* $0, 0,\ldots, 0$.

Proof. We have

$$\sum_{a_1,\ldots,a_r} |T(a_1,\ldots,a_r)|^2 = p^r N, \tag{3}$$

where N is the number of solutions of

$$\begin{gathered} x_1+\ldots+x_r = x_1'+\ldots+x_r' \\ x_1x_2+\ldots = x_1'x_2'+\ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_1\ldots x_r = x_1'\ldots x_r'. \end{gathered}$$

These conditions imply that the x' are a permutation of the x, hence $N = O(p^r)$. In the sum (3) there occur at least $(p-1)/r$ different sets of a's equivalent to any given set of a's other than $0,\ldots, 0$. Hence the result.

LEMMA 2. *If ϑ is any generating element of $[p^k]$, and χ is any non-principal character of $[p^k]$, then*

$$\sum_x \chi(x+\vartheta) = O\left(p^{1-\frac{1}{2(k+1)}}\right).$$

Proof. The sum on the left is unaltered when ϑ is replaced by $\vartheta+t$. Hence we can suppose without loss of generality that the equation satisfied by ϑ has the form

$$\vartheta^k = c_{k-1}\vartheta^{k-1}+\ldots+c_0, \quad \text{where} \quad c_j \neq 0 \quad (j = 0, 1,\ldots, k-1). \tag{4}$$

For, if one of the c's is zero in the equation satisfied by $\vartheta+t$, then t must satisfy an equation (not identically zero) whose degree depends only on k, and, since k is fixed and p large, this cannot happen for all t.

Denoting by S the sum in question, we have, for any positive integer r,

$$S^r = \sum_{x_1,\ldots,x_r} \chi\{(x_1+\vartheta)\ldots(x_r+\vartheta)\}.$$

Hence, by (1),

$$\begin{aligned} S^r &= \frac{1}{\tau(\bar\chi)} \sum_\xi \bar\chi(\xi) \sum_{x_1,\ldots,x_r} e(\operatorname{Sp}\xi(x_1+\vartheta)\ldots(x_r+\vartheta)) \\ &= \frac{1}{\tau(\bar\chi)} \sum_\xi \bar\chi(\xi) e(\operatorname{Sp}\xi\vartheta^r) T(\operatorname{Sp}\xi\vartheta^{r-1}, \operatorname{Sp}\xi\vartheta^{r-2},\ldots, \operatorname{Sp}\xi). \end{aligned}$$

By Cauchy's inequality,

$$|S|^{2r} \leqslant |\tau(\bar{\chi})|^{-2} \sum_{\xi} 1 \sum_{\xi \neq 0} |T(\operatorname{Sp} \xi \vartheta^{r-1}, \ldots, \operatorname{Sp} \xi)|^2$$
$$= \sum_{\xi \neq 0} |T(\operatorname{Sp} \xi \vartheta^{r-1}, \ldots, \operatorname{Sp} \xi)|^2,$$

by (2).

We now choose $r = k+1$. We prove that, for any given $\xi \neq 0$, there are at most $O(1)$ values of ξ' for which $\operatorname{Sp} \xi \vartheta^k, \operatorname{Sp} \xi \vartheta^{k-1}, \ldots, \operatorname{Sp} \xi$ and $\operatorname{Sp} \xi' \vartheta^k, \operatorname{Sp} \xi' \vartheta^{k-1}, \ldots, \operatorname{Sp} \xi'$ are equivalent. The equivalence of these two sets implies the existence of a $u \neq 0$ such that

$$u^{k+1-j} \operatorname{Sp} \xi \vartheta^j = \operatorname{Sp} \xi' \vartheta^j \quad (j = 0, 1, \ldots, k).$$

Then, by (4),

$$u \operatorname{Sp} \xi \vartheta^k = \operatorname{Sp} \xi' \vartheta^k = c_{k-1} \operatorname{Sp} \xi' \vartheta^{k-1} + \ldots + c_0 \operatorname{Sp} \xi'$$
$$= c_{k-1} u^2 \operatorname{Sp} \xi \vartheta^{k-1} + \ldots + c_0 u^{k+1} \operatorname{Sp} \xi.$$

For given $\xi \neq 0$, this is an equation in u not all of whose coefficients vanish, since $c_j \neq 0$ for any j and not all of $\operatorname{Sp} \xi \vartheta^{k-1}, \ldots, \operatorname{Sp} \xi$ are zero. Hence there are not more than $O(1)$ values for u, and u determines ξ' uniquely.

From this, and from the fact that for $\xi \neq 0$ not all of $\operatorname{Sp} \xi \vartheta^k, \ldots, \operatorname{Sp} \xi$ vanish, we have

$$|S|^{2r} \leqslant O(1) \sum_{a_1, \ldots, a_r}^{*} |T(a_1, \ldots, a_r)|^2$$
$$= O(p^{2r-1}),$$

i.e.

$$S = O(p^{1-(1/2r)}) = O\left(p^{1-\frac{1}{2(k+1)}}\right).$$

Proof of Theorem 1. The proof follows that of Vinogradov's result given by Landau (loc. cit.). An element α of $[p^k]$ is a primitive root of $[p^k]$ if and only if

$$\sum_{d \mid p^k - 1} \frac{\mu(d)}{\phi(d)} \sum_{\chi^{(d)}} \chi^{(d)}(\alpha) \neq 0,$$

where in the inner sum $\chi^{(d)}$ runs through the $\phi(d)$ characters of $[p^k]$ whose exact order is d. Hence, if no element of the form $\vartheta + x$ is a primitive root, we should have

$$\sum_{d \mid p^k - 1} \frac{\mu(d)}{\phi(d)} \sum_{\chi^{(d)}} \sum_{x} \chi^{(d)}(\vartheta + x) = 0.$$

The contribution of $d = 1$ is p, and the contribution of any other d does not exceed

$$\frac{1}{\phi(d)} \sum_{\chi^{(d)}} O\left(p^{1-\frac{1}{2(k+1)}}\right) = O\left(p^{1-\frac{1}{2(k+1)}}\right).$$

Hence we should have

$$p = O\left(d(p^k-1)p^{1-\frac{1}{2(k+1)}}\right),$$

which is false for $p > p_0(k)$.

Proof of Theorem 2

LEMMA 3. *If $M(x)$, $A(x)$ are given polynomials* (mod p) *which are relatively prime, then there exists an infinity of irreducible polynomials $P(x)$ of even degree with highest coefficient* 1 *which satisfy*

$$P(x) \equiv A(x) \pmod{M(x)}.$$

This is a particular case of a result proved by Kornblum.†

Proof of Theorem 2. By Lemma 3 there exists an infinity of irreducible $P(x)$ of even degree with highest coefficient 1 which satisfy

$$P(x) \equiv 1 \pmod{x^p-x}. \tag{5}$$

Let $k = 2k'$ be the degree of such a polynomial $P(x)$, and let ϑ be a root of $P(x)$. For all integers a, b ($a \not\equiv 0 \pmod p$), we have, on writing $b \equiv -ac \pmod p$,

$$N(a\vartheta+b) = \prod_j (a\vartheta^{(j)}-ac) = (-a)^{2k'} \prod_j (c-\vartheta^{(j)})$$
$$= (-a)^{2k'}P(c),$$

where the product is extended over the roots $\vartheta^{(j)}$ of $P(x)$, and $N(\alpha)$ denotes the norm of an element α of $[p^k]$. By (5), $P(c) \equiv 1 \pmod p$. Hence, for any a, b, $N(a\vartheta+b)$ is a quadratic residue (mod p). Thus $a\vartheta+b$ is a quadratic residue in $[p^k]$,‡ and so is not a primitive root of $[p^k]$.

The case $p = 2$ has been omitted for the sake of simplicity. It can be dealt with in a similar way by proving that there exist polynomials $Q(x)$, irreducible over $[2^2]$, which satisfy $Q(x) \equiv 1$ $(\text{mod}\,(x^4-x))$. Then, if ϑ is a root of $Q(x)$, it is easily seen that ϑ, $\vartheta+1$ are each cubic residues in $[2^{2k}]$.

† *Math. Zeits.* 5 (1919), 100–11, § 2.

‡ If $(N\alpha)^{\frac{1}{2}(p-1)} = 1$, then $\alpha^{\frac{1}{2}(p^k-1)} = 1$, since $N\alpha = \alpha^{1+p+\ldots+p^{k-1}} = \alpha^{(p^k-1)/(p-1)}$.

ON SOME INFINITE SERIES INVOLVING ARITHMETICAL FUNCTIONS (II)

By H. DAVENPORT (*Cambridge*)

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1. In a previous paper with this title* some formal identities involving infinite series were considered, of which the most interesting was

$$\sum_{1}^{\infty} \frac{\mu(n)}{n}\{n\theta\} = -\frac{1}{\pi}\sin 2\pi\theta, \tag{A}$$

where $\{t\} = t-[t]-\frac{1}{2}$. It was observed that the series converges to the sum stated for rational θ, and it was proved that this is true for almost all θ. It was also proved that the partial sums of the series are uniformly bounded.

In this paper, using deeper methods based on Vinogradov's recent work on the theory of primes,† I shall prove that the series in fact converges uniformly in θ to the sum stated.

The greater part of the paper is concerned with the proof that, for any fixed h,

$$\sum_{n=1}^{x} \mu(n)e^{2\pi in\theta} = O(x(\log x)^{-h}) \tag{1}$$

uniformly in θ. The application of this result to our problem requires only a natural modification of the method of (**I**).

Throughout the paper c_1, c_2,... denote positive absolute constants, and $C_1(\epsilon)$,... denote positive numbers depending only on the variable specified; p always denotes a prime. The constants implied by the symbol O depend only on ϵ, h, h_1, H, K.

2. Lemmas 1–4 are concerned with the Dirichlet L-functions (mod q), where q is any positive integer, and use the notation common in the subject. The line of argument adopted (suggested to me by Dr. Heilbronn) is designed simply to establish Lemma 4 with the minimum of trouble, making full use of the work of Titchmarsh‡ and Page.§

* See above, pp. 8–13. This will be referred to as (**I**).

† *Recueil Mathématique*, 2 (1937), 179–94. This will be referred to as **V**.

‡ *Rend. di Palermo*, 54 (1930), 414–29, with correction in *Rend. di Palermo*, 57 (1933), 478–9. This will be referred to as **T**.

§ *Proc. London Math. Soc.* 39 (1935), 116–41. This will be referred to as **P**.

LEMMA 1. *There exists c_1 such that, for $|t| \geqslant 4$, $\sigma \geqslant 1-c_1/(\log q|t|)$,*

$$\frac{L'(s,\chi)}{L(s,\chi)} = O(\log q|t|).$$

This is Lemma 4 of **T**.

LEMMA 2. *For any $\epsilon > 0$ there exists $C_1(\epsilon)$ such that $L(s,\chi) \neq 0$ for $|t| \leqslant 5$, $\sigma \geqslant 1-C_1(\epsilon)q^{-\epsilon}$ and non-principal χ.*

Proof. If χ is a complex character, the result follows from Lemma 6 of **T**. Suppose now that χ is a real non-principal character. By Lemma 6 of **P**, we can suppose that the only possible zero of $L(s,\chi)$ in the domain in question is real. It suffices to consider only primitive characters. Then there exists $\Delta = \pm q$ such that

$$L(s,\chi) = L_\Delta(s) = \sum_{n=1}^{\infty} \left(\frac{\Delta}{n}\right) n^{-s},$$

where (Δ/n) is the Kronecker symbol. The result now follows from Siegel's famous inequality*

$$L_\Delta(1) > C_2(\epsilon)|\Delta|^{-\epsilon}$$

and the elementary inequality

$$L'_\Delta(s) = O(\log^2|\Delta|) \quad \text{for} \quad s \geqslant 1-(\log|\Delta|)^{-1}.$$

LEMMA 3. *For any $\epsilon > 0$ there exists $C_3(\epsilon)$ such that, for $|t| \leqslant 4$ and $\sigma \geqslant 1-C_3(\epsilon)q^{-\epsilon}$, and non-principal χ,*

$$\frac{L'(s,\chi)}{L(s,\chi)} = O(q^\epsilon).$$

Proof. Lemma 2 above and Lemma 10 of **T**.

For $\epsilon > 0$ let

$$C_4(\epsilon) = \min\left(C_3(\epsilon), c_1 \min_{q=1,2,\ldots} \frac{q^\epsilon}{\log 4q}\right).$$

Let Γ be the path

$$\sigma = 1-C_4(\epsilon)q^{-\epsilon} \quad (|t| < 4),$$

$$\sigma = 1-\frac{c_1}{\log q|t|} \quad (|t| > 4),$$

$$1-\frac{c_1}{\log 4q} \leqslant \sigma \leqslant 1-C_4(\epsilon)q^{-\epsilon} \quad (|t| = 4).$$

* *Acta Arithmetica*, 1 (1936), 83–6.

LEMMA 4. *On Γ we have*

$$\frac{1}{L(s,\chi)} = O(\log q|t|) \quad (|t| > 4),$$
$$\frac{1}{L(s,\chi)} = O(q^{\epsilon}) \quad (|t| \leqslant 4).$$

Proof. If χ is the principal character, the results follow from well-known properties of the ζ-function. We suppose therefore that χ is non-principal. Let $s_1 = \sigma_1 + it_1$ be a point on Γ, and let

$$s_2 = 2-\sigma_1+it_1.$$

Lemma 1 (for $|t| \geqslant 4$) and Lemma 3 (for $|t| < 4$) tell us that, on the straight line joining s_1 and s_2, we have

$$\frac{L'(s,\chi)}{L(s,\chi)} = O\left(\frac{1}{1-\sigma_1}\right).$$

Hence

$$\left|\log\frac{L(s_2,\chi)}{L(s_1,\chi)}\right| = \left|\int\limits_{s_1}^{s_2} \frac{L'(s,\chi)}{L(s,\chi)}\,ds\right| = O\left(|s_2-s_1|\frac{1}{1-\sigma_1}\right) = O(1).$$

It follows that $$\frac{1}{L(s_1,\chi)} = O\left(\left|\frac{1}{L(s_2,\chi)}\right|\right).$$

Since the Dirichlet series for $1/L(s_2,\chi)$ is majorized by $\sum\limits_{1}^{\infty} n^{-2+\sigma_1}$, we have

$$\frac{1}{L(s_1,\chi)} = O\left(\frac{1}{1-\sigma_1}\right),$$

which proves the Lemma.

LEMMA 5. *If $(l,q) = 1$, and $q \leqslant (\log x)^h$, then*

$$\sum_{\substack{n=1 \\ n\equiv l \,(\mathrm{mod}\, q)}}^{x} \mu(n) = O(xe^{-C_5(h)\sqrt{(\log x)}}).$$

Proof. The proof is by the usual contour integration, as in **P** (§ 4), but with $1/L(s,\chi)$ instead of $L'(s,\chi)/L(s,\chi)$ and with the path Γ, with $|t| \leqslant T = e^{\sqrt{(\log x)}}$, as path of integration. The proof is slightly simplified by the fact that $1/L(s,\chi)$ has no pole to the right of Γ. The contribution of the part of Γ with $|t| > 4$ is, as in **P**,

$$O(xe^{-c_2\sqrt{(\log x)}}).$$

The contribution of the part of Γ with $|t| \leqslant 4$ is

$$O(q^{\epsilon}x^{1-C_4(\epsilon)q^{-\epsilon}}) = O((\log x)^{h\epsilon}xe^{-C_4(\epsilon)(\log x)^{1-h\epsilon}}) = O(xe^{-C_5(h)\sqrt{(\log x)}}),$$

on choosing $\epsilon = (3h)^{-1}$. Hence we obtain the result stated.

LEMMA 6. *The condition $(l,q) = 1$ in Lemma 5 can be omitted.*

Proof. Suppose $l = \delta l'$, $q = \delta q'$, where $\delta > 1$ and $(l',q') = 1$. Then

$$\sum_{\substack{n=1\\ n\equiv l\,(\mathrm{mod}\,q)}}^{x} \mu(n) = \mu(\delta) \sum_{\substack{n\leqslant x/\delta\\ n\equiv l'\,(\mathrm{mod}\,q')\\ (n,\delta)=1}} \mu(n).$$

Without loss of generality we can suppose δ *quadratfrei*, and we can replace the condition $(n,\delta) = 1$ by $(n,\delta_1) = 1$, where $\delta_1 = \delta/(\delta,q')$, so that $(\delta_1,q') = 1$. The sum on the right can now be expressed as the sum of $\phi(\delta_1)$ sums each of the type of that in Lemma 5, with x replaced by x/δ and q replaced by $\delta_1 q'$. Since $\delta_1 q' \leqslant q$ and $\delta \leqslant q \leqslant (\log x)^h$, the hypothesis of Lemma 5 is satisfied with a different value of h, and the required result follows with a different value for $C_5(h)$.

In what follows $H > 0$ is fixed, and

$$\tau = [x(\log x)^{-H}].$$

LEMMA 7. *If $|\theta - a/q| \leqslant q^{-1}\tau^{-1}$, where $(a,q) = 1$, and $q \leqslant (\log x)^H$, then*

$$\sum_{n=1}^{x} \mu(n) e^{2\pi i n\theta} = O(xe^{-C_6(H)\sqrt{(\log x)}}).$$

Proof. Let $S_0 = 0$, and for $1 \leqslant n \leqslant x$ let

$$S_n = \sum_{m=1}^{n} \mu(m) e^{2\pi i m a/q}$$

$$= \sum_{r=1}^{q} e^{2\pi i a r/q} \sum_{\substack{m=1\\ m\equiv r\,(\mathrm{mod}\,q)}}^{n} \mu(m).$$

By Lemma 6,

$$S_n = O(qxe^{-C_7(H)\sqrt{(\log x)}})$$
$$= O(xe^{-C_8(H)\sqrt{(\log x)}}).$$

By partial summation, writing $\theta = a/q + \beta$, we have

$$\sum_{n=1}^{x} \mu(n) e^{2\pi i n(a/q+\beta)} = \sum_{n=1}^{x} (S_n - S_{n-1}) e^{2\pi i n\beta}$$

$$= \sum_{n=1}^{x} S_n\, e^{2\pi i n\beta}(1 - e^{2\pi i\beta}) + S_x\, e^{2\pi i\beta(x+1)}$$

$$= O\big((x|1 - e^{2\pi i\beta}| + 1) xe^{-C_8(H)\sqrt{(\log x)}}\big)$$

$$= O\big((xq^{-1}\tau^{-1} + 1) xe^{-C_8(H)\sqrt{(\log x)}}\big)$$

$$= O(xe^{-C_6(H)\sqrt{(\log x)}}).$$

3. LEMMA 8.* *Let N be a positive integer, $1 < U_0 < U_1 < N$, $1 \leqslant q \leqslant N$, $(a, q) = 1$. Let $\theta_1(x, N)$, $\theta_2(y, N)$ be arbitrary functions, each of which is absolutely bounded. Let $\psi(y, N)$ be an arbitrary function. Then*

$$\sum_{U_0 < x \leqslant U_1} \theta_1(x, N) \sum_{\substack{1 \leqslant y \leqslant N/x \\ \psi(y,N) < x}} \theta_2(y, N) e^{2\pi i a x y/q}$$
$$= O\left\{N(\log N)^2 \sqrt{\left(\frac{1}{U_0} + \frac{U_1}{N} + \frac{1}{q} + \frac{q}{N}\right)}\right\}.$$

Proof. It suffices to prove that, for $1 < U < N$, $U < U' \leqslant 2U$,

$$\sum_{U < x \leqslant U'} \theta_1(x, N) \sum_{\substack{1 \leqslant y \leqslant N/x \\ \psi(y,N) < x}} \theta_2(y, N) e^{2\pi i a x y/q}$$
$$= O\left\{N \log N \sqrt{\left(\frac{1}{U} + \frac{U}{N} + \frac{1}{q} + \frac{q}{N}\right)}\right\}, \tag{2}$$

since the sum of the Lemma can be split up into $O(\log N)$ such sums. The square of the absolute value of the left-hand side of (2) does not exceed

$$U \sum_{U < x \leqslant 2U} |\theta_1(x, N)|^2 \left| \sum_{\substack{1 \leqslant y \leqslant N/x \\ \psi(y,N) < x}} \theta_2(y, N) e^{2\pi i a x y/q} \right|^2$$
$$= O\left(U \sum_{U < x \leqslant 2U} \sum_{\substack{1 \leqslant y_1 \leqslant N/x \\ \psi(y_1,N) < x}} \sum_{\substack{1 \leqslant y_2 \leqslant N/x \\ \psi(y_2,N) < x}} \theta_2(y_1, N) \overline{\theta_2(y_2, N)} e^{2\pi i a x (y_1 - y_2)/q}\right)$$
$$= O\left\{U \sum_{1 \leqslant y_1 \leqslant N/U} \sum_{1 \leqslant y_2 \leqslant N/U} \min\left(U, \frac{1}{\langle a(y_1 - y_2)/q \rangle}\right)\right\}.$$

where $\langle t \rangle$ denotes the distance of t from the nearest integer. The above expression is

$$O\left\{U \frac{N}{U} \sum_{0 \leqslant z \leqslant N/U} \min\left(U, \frac{1}{\langle az/q \rangle}\right)\right\} = O\left\{N(U + q \log q)\left(\frac{N}{Uq} + 1\right)\right\}$$
$$= O\left\{N^2 \log N \left(\frac{1}{U} + \frac{U}{N} + \frac{1}{q} + \frac{q}{N}\right)\right\}.$$

This establishes (2).

LEMMA 9. *If $h_1 > 3$, $(\log N_1)^{3h_1} < q_1 \leqslant N_1 (\log N_1)^{-3h_1}$, $(b, q_1) = 1$, then*

$$\sum_{p \leqslant N_1} e^{2\pi i b p/q_1} = O(N_1 (\log N_1)^{2 - h_1}).$$

This is a special case of Theorem 1 of V.

* This is a slight variation on Vinogradov's main Lemma (Lemma 4 of V).

LEMMA 10. *If* $h > 1$, $(\log N)^{12h} < q < N(\log N)^{-12h}$, $(a,q) = 1$, *then*

$$\sum_{n=1}^{N} \mu(n)e^{2\pi i a n/q} = O(N(\log N)^{2-h}).$$

Proof. Let $\psi(n)$ denote the greatest prime factor of n. If a *quadratfrei* number n with $\sqrt{N} \leqslant n \leqslant N$ has $\psi(n) \leqslant (\log N)^{2h}$, then n has at least $\frac{1}{2}\log N/(2h \log\log N)$ prime factors, so that

$$d(n) \geqslant 2^{\log N/(4h \log\log N)} > (\log N)^h,$$

for sufficiently large N. Hence

$$\begin{aligned}\sum_{\substack{n=2\\ \psi(n)\leqslant(\log N)^{2h}}}^{N} \mu(n)e^{2\pi i a n/q} &= O\Big(\sum_{n=1}^{N} d(n)(\log N)^{-h}\Big)+O(\sqrt{N})\\ &= O(N(\log N)^{1-h}).\end{aligned}$$

We have

$$\begin{aligned}&\sum_{\substack{n=2\\ \psi(n)>(\log N)^{2h}}}^{N} \mu(n)e^{2\pi i a n/q}\\ &\quad= \sum_{(\log N)^{2h}<p\leqslant N}\ \sum_{\substack{m\leqslant N/p\\ \psi(m)<p}} \mu(pm)e^{2\pi i a p m/q}\\ &\quad= -\sum_{(\log N)^{2h}<p\leqslant N}\ \sum_{\substack{m\leqslant N/p\\ \psi(m)<p}} \mu(m)e^{2\pi i a p m/q}\\ &\quad= -\sum_{m\leqslant(\log N)^{2h}} \mu(m) \sum_{(\log N)^{2h}<p\leqslant N/m} e^{2\pi i a m p/q} -\\ &\qquad\quad -\sum_{(\log N)^{2h}<p<N(\log N)^{-2h}}\ \sum_{\substack{(\log N)^{2h}<m\leqslant N/p\\ \psi(m)<p}} \mu(m)e^{2\pi i a p m/q}\\ &\quad= -S_1-S_2, \quad \text{say.}\end{aligned}$$

The inner sum in S_1 satisfies the conditions of Lemma 9, with

$$q_1 = \frac{q}{(m,q)}, \qquad b = \frac{am}{(m,q)}, \qquad N_1 = \frac{N}{m}, \qquad h_1 = 3h,$$

hence

$$\begin{aligned}S_1 &= O((\log N)^{2h}N(\log N)^{2-3h})\\ &= O(N(\log N)^{2-h}).\end{aligned}$$

The sum S_2 is that of Lemma 8, with $\theta_1(x,N) = 1$ if x is a prime, otherwise 0; and $\theta_2(y,N) = \mu(y)$ if $y > (\log N)^{2h}$, otherwise 0; and $\psi(y,N) = \psi(y)$. Hence, by Lemma 8,

$$S_2 = O(N(\log N)^{2-h}).$$

LEMMA 11. *If* $H > 14$, $|\theta - a/q| \leqslant q^{-1}\tau^{-1}$, *where* $(a,q) = 1$ *and* $(\log x)^H < q \leqslant \tau$, *then*

$$\sum_{n=1}^{x} \mu(n)e^{2\pi i n\theta} = O(x(\log x)^{2-\frac{1}{14}H}).$$

Proof. By the same partial summation as that used in the proof of Lemma 7, noting that

$$x|1-e^{2\pi i\beta}|+1 = O(xq^{-1}\tau^{-1}+1) = O(1),$$

it suffices to prove that, for $N \leqslant x$,

$$\sum_{n=1}^{N} \mu(n)e^{2\pi i a n/q} = O\big(x(\log x)^{2-h}\big) \quad (h = \tfrac{1}{14}H).$$

If $N \leqslant x(\log x)^{-h}$ this is trivial, and, if $x(\log x)^{-h} < N \leqslant x$, the hypotheses of Lemma 10 are satisfied, and the result follows.

4. THEOREM 1. *For any given K,*

$$\sum_{n=1}^{x} \mu(n)e^{2\pi i n\theta} = O\big(x(\log x)^{-K}\big),$$

uniformly in θ.

Proof. Choose H such that $2-\frac{1}{14}H < -K$. There exist integers a, q with $(a,q) = 1$, $1 \leqslant q \leqslant \tau$, such that $|\theta-a/q| \leqslant q^{-1}\tau^{-1}$. If $q \leqslant (\log x)^{H}$ the result follows from Lemma 7, and, if $(\log x)^{H} < q \leqslant \tau$, it follows from Lemma 11.

5. We now come to the application of Theorem 1 to the series (A).

LEMMA 12. *If $q \leqslant (\log x)^{h}$, then*

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{x} \frac{\mu(n)}{n} = O\big((\log x)^{-h}\big).$$

Proof. We have

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{x} \frac{\mu(n)}{n} = -\sum_{\substack{n>x \\ (n,q)=1}} \frac{\mu(n)}{n} = -\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{n>x \\ n\equiv a \,(\mathrm{mod}\, q)}} \frac{\mu(n)}{n}.$$

By partial summation from Lemma 5, the inner sum is

$$O\big(e^{-C_9(h)\sqrt{(\log x)}}\big),$$

whence the result.

Let $$R_N(\theta) = \sum_{n=1}^{N} \frac{\mu(n)}{n}\{n\theta\} + \frac{1}{\pi}\sin 2\pi\theta.$$

LEMMA 13. *For arbitrary N, θ_1, θ_2, and any fixed h, we have*

$$R_N(\theta_2)-R_N(\theta_1) = O(N|\theta_2-\theta_1|)+O\big((\log N)^{-h}\big).$$

Proof. The proof is similar to that of Lemma 2 of (I). The discontinuity of $R_N(\theta)$ at a Farey point a/q of order N is

$$-\sum_{\substack{n=1 \\ q|n}}^{N} \frac{\mu(n)}{n} = -\frac{\mu(q)}{q} \sum_{\substack{n\leqslant N/q \\ (n,q)=1}} \frac{\mu(n)}{n}.$$

By Lemma 1 of (**I**) this is $O(1/q)$, and by Lemma 12 above, if $q \leqslant (\log N)^h$, it is $O((\log N)^{-h})$. Thus in the proof of Lemma 2 of (**I**) we obtain the upper bound

$$O(N|\theta_1-\theta_2|)+O((\log N)^{-h})$$

for the sum of the discontinuities of $R_N(\theta)$ in the interval (θ_1, θ_2), whence the result.

With the notation of (**I**), let

$$T_N(\theta) = \sum_{n>N} \frac{\mu(n)}{n^2}\{n\theta_1\},$$

so that

$$\int_{\theta_1}^{\theta_2} R_N(\theta)\, d\theta = T_N(\theta_2)-T_N(\theta_1).$$

Lemma 14. *For any given h, $T_N(\theta) = O(N^{-1}(\log N)^{-h})$, uniformly in θ.*

Proof. We have

$$T_N(\theta) = \frac{1}{2\pi^2}\sum_{n>N} \mu(n)n^{-2} \sum_{m=1}^{\infty} m^{-2}\cos 2\pi mn\theta$$

$$= \frac{1}{2\pi^2}\sum_{m=1}^{\infty} m^{-2} \sum_{n>N} \mu(n)n^{-2}\cos 2\pi mn\theta.$$

Hence the result, by partial summation from Theorem 1.

Theorem 2. *The series*

$$\sum_{1}^{\infty} \frac{\mu(n)}{n}\{n\theta\}$$

converges uniformly in θ to the sum $-(1/\pi)\sin 2\pi\theta$. More precisely, $R_N(\theta) = O((\log N)^{-h})$ for any given h, uniformly in θ.

Proof. For arbitrary θ_1 let $\theta_2 = \theta_1+N^{-1}(\log N)^{-h}$. By Lemma 14,

$$\int_{\theta_1}^{\theta_2} R_N(\theta)\, d\theta = O(N^{-1}(\log N)^{-2h}),$$

uniformly in θ_1. By Lemma 13,

$$\int_{\theta_1}^{\theta_2} R_N(\theta)\, d\theta = (\theta_2-\theta_1)R_N(\theta_1)+O((\theta_2-\theta_1)\{N(\theta_2-\theta_1)+(\log N)^{-h}\})$$

$$= N^{-1}(\log N)^{-h}R_N(\theta_1)+O(N^{-1}(\log N)^{-2h}).$$

This proves the theorem.

Exactly the same arguments apply to

$$\sum_{1}^{\infty} \frac{\lambda(n)}{n}\{n\theta\} = -\frac{1}{\pi}\sum_{1}^{\infty} \frac{\sin 2n^2\pi\theta}{n^2}. \tag{B}$$

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